(CONSTRUCTION OF A VECTOR ⊥ TO A SYSTEM OF GIVEN VECTORS.)

Given a vector $a = a_1 i + a_2 j$ Const a vector ⊥ to a. we shall denote it by $\breve{a}$ and it is:

(1) $$a = a_1 i + a_2 j$$
(2) $$\breve{a} = -a_2 i + a_1 j$$

(2) is obtained from (1) by taking the column cofactors of (1) with their signs changed. (2) is the left normal of (1). It will be observed from (1) and (2) that $a_0 = \breve{a}_0$.

Given two vectors a and b to construct a vector c ⊥ to a and b.

(3) $$a = a_1 i + a_2 j + a_3 k$$
(4) $$b = b_1 i + b_2 j + b_3 k$$

(5) $$c = c_1 i + c_2 j + c_3 k$$

(6) $$c = (a \times b)$$
(7) $$c = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k$$

Sept. 29, 1959

(6) is identically ⊥ to (3) and (4).
(7) is gotten from (3) and (4) by taking the column cofactors from left to right of the system (3-4). This construction is important. Note the Column Cofactor notion.

Given three vectors a, b, c to construct a vector d ⊥ to a, b, and c.

(8) $$a = a_{11} i_1 + a_{12} i_2 + a_{13} i_3 + a_{14} i_4$$

(9) $$b = a_{21} i_1 + a_{22} i_2 + a_{23} i_3 + a_{24} i_4$$

(10) $$c = a_{31} i_1 + a_{32} i_2 + a_{33} i_3 + a_{34} i_4$$

(11) $$\mathbf{d} = d_1 i_1 + d_2 i_2 + d_3 i_3 + d_4 i_4$$

(12) $$d_1 = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$$

(13) $$d_2 = - \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$$

(14) $$d_3 = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$$

(15) $$d_4 = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

One notes that the d_n are the column cofactors of the system (8-10). This is very important. It may be shown that one may construct a vector of N components that will be ⊥ to N-1 vectors of N components. The N components will be the N column cofactors of the N-1 vectors of N components.

This last fact is most important for it enables one to diagonalize a matrix beautifully. We give an illustrative example of each item:

Oct. 1 1959

Examp 1

Given

(1) $a = 3i + 2j$

(2) $\breve{a} = -2i + 3j$ (Left normal)

Ex. 2

Given

(3) $a = 3i + 2j + 1k$

(4) $b = 1i + 4j + 5k$

(5) $c = (a \times b) = 3i - 7j + 5k$

that (5) is ⊥ to both (3) and (4) may be seen by multiplication. the coefficients in (5) are the column cofactors of (3-4) taken in order with the common factor 2 cancelled out.

Oct. 1 1959

EX. 3

(6) $a = 2i_1 + 3i_2 + 4i_3 + 1i_4$

(7) $b = 1i_1 + 2i_2 - 3i_3 - 2i_4$

(8) $c = -1i_1 - 3i_2 + 1i_3 + 3i_4$

(9) $d = -14i_1 + 10i_2 - 2i_3 + 6i_4$

or

(10) $(d = -7i_1 + 5i_2 - 1i_3 + 3i_4)$

The coefficients in (10) are the column cofactors of (6-8) in order with the common factor 2 cancelled from them. That (10) is ⊥ to (6) (7) and (8) may be seen by multiplying each by (10):

(11) $d \cdot a = -14 + 15 - 4 + 3 = 0$

and in the same way the others:

(12) $d \cdot b = d \cdot c = 0$

This is a beautiful thing.

Oct. 1 1959

Suppose that we have a system of equations:

(13) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdots & \cdots & \cdots \quad \cdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots \\ X_n \end{vmatrix} = \begin{vmatrix} P_1 \\ P_2 \\ P_3 \\ \cdots \\ P_n \end{vmatrix}$$

we may multiply (13) by a matrix

(14) $$\begin{vmatrix} b_{11} & b_{12} & b_{13} \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} \cdots & b_{3n} \\ \cdots & \cdots & \cdots \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} \cdots & b_{nn} \end{vmatrix}$$

getting:

(15) $$\begin{vmatrix} b_{11} & \\ & b_{nn} \end{vmatrix} \begin{vmatrix} a_{11} & \\ & a_{nn} \end{vmatrix} \begin{vmatrix} X_1 \\ X_n \end{vmatrix} = \begin{vmatrix} c_{11} & \\ & c_{nn} \end{vmatrix} \begin{vmatrix} X_1 \\ X_n \end{vmatrix} = \begin{vmatrix} c_{11} & \\ & c_{nn} \end{vmatrix} \begin{vmatrix} P_1 \\ P_n \end{vmatrix} = \begin{vmatrix} S_1 \\ S_n \end{vmatrix}$$

Oct. 1, 1959

If the $|c_{11} \; c_{nn}|$ is to be diagonal we must construct our row vector: $(b_{11} \; b_{12} \; b_{13} \cdots b_{1n}) \perp$ to the column vectors:

$$
\begin{array}{lll}
a_{12}, & a_{13}, \cdots & a_{1n} \\
a_{22}, & a_{23}, \cdots & a_{2n} \\
a_{32}, & a_{33}, \cdots & a_{3n} \\
'' & '' & '' \\
a_{n2} & a_{n3}, \cdots & a_{nn}
\end{array} \qquad (15)
$$

and $(b_{21} \; b_{22} \; b_{23} \cdots b_{2n}) \perp$ to the column vectors:

$$
\begin{array}{lll}
a_{11}, & a_{13}, \cdots & a_{1n} \\
a_{21}, & a_{23}, \cdots & a_{2n} \\
a_{31}, & a_{33}, \cdots & a_{3n} \\
'' & '' & '' \\
a_{n1}, & a_{n3}, \cdots & a_{nn}
\end{array} \qquad (16)
$$

and $(b_{n1} \; b_{n2} \; b_{n3} \cdots b_{nn}) \perp$ to the column vectors:

Oct, 1, 1959

$$
(17) \quad
\begin{matrix}
a_{11} & a_{12} \cdots & a_{1(r-1)} & a_{1(r+1)} & a_{1(r+2)} \cdots & a_{1n} \\
a_{21} & a_{22} \cdots & a_{2(r-1)} & a_{2(r+1)} & a_{2(r+2)} \cdots & a_{2n} \\
a_{31} & a_{32} \cdots & a_{3(r-1)} & a_{3(r+1)} & a_{3(r+2)} \cdots & a_{3n} \\
\vdots & \vdots & \cdots & \vdots & \vdots \quad \vdots & \vdots \\
a_{n1} & a_{n2} \cdots & a_{n(r-1)} & a_{n(r+1)} & a_{n(r+2)} \cdots & a_{nn}
\end{matrix}
$$

Our problem here is to construct a vector of n dimensions $\perp$ to $(n-1)$ vectors of n dimensions. To construct the components of the last row vector we take the row cofactors of (17) from top to bottom in order for its components, cancelling any common factor.

EX. 4

Suppose we have the system of equations on the following page: it is a system of three numerical equations diagonalized then solved mutationwise:

Oct. 2, 1959

EX. 4

48

$$\begin{vmatrix} 2 & +3 & +1 \\ 1 & +1 & -1 \\ 3 & -1 & -2 \end{vmatrix} r = \begin{vmatrix} 4 \\ -2 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} -3 & +5 & -4 \\ 1 & +7 & -3 \\ -4 & +11 & -1 \end{vmatrix} \begin{vmatrix} 2 & +3 & +1 \\ 1 & +1 & -1 \\ 3 & -1 & -2 \end{vmatrix} r = \begin{vmatrix} -3 & +5 & -4 \\ 1 & +7 & -3 \\ -4 & +11 & -1 \end{vmatrix} \begin{vmatrix} 4 \\ -2 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} -13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & -13 \end{vmatrix} r = \begin{vmatrix} -26 \\ -13 \\ -39 \end{vmatrix}$$

$$(x, y, z) = (2, -1, 3)$$

MUTATION VIEW

$$\begin{matrix} 4 & 5 & -1 \\ 7 & -1 & -5 \end{matrix}$$

$$(-26, 13, -39)$$
$$(-2, 1, -3)$$

$$(2, -1, 3) = (x, y, z)$$