Matrix Inversion Thru The Eigen-Value Equation

The eigen-equation of a matrix A of order n may be written

$$(1) \qquad A \cdot r = S r$$

Here S is the eigen-value of the matrix A. We may write (1) as

$$(2) \qquad (A - S I) \cdot r = 0$$

r in (2) can have significant values only when the determinant of the matrix in (2) is zero. Setting this equal to zero and expanding one gets an equation of the form:

$$(3) \qquad S^{n} - T_1 S^{n-1} + T_2 S^{n-2} - T_3 S^{n-3} + \dots\dots + (-1)^{n} T_n = 0$$

A REVELATION

In the unpublished Mutation Geometry (see the abstract of a paper presented before the Ohio Section of the American Mathematical Association meeting in Miami University, Oxford , Ohio. See the American Mathematical Monthly, Aug.- Sept. issue page 645(1959), "The Mutation Geometry View of the Conics") it is shown that the T_j is the jth order trace of the matrix and that the elements in the trace follow the binomial law. This is quite a revelation, a fact not appearing in the literature. We claim its origin is in the new Mutation Geometry, the Science of Intangible Change.

It is known that the matrix satisfies its own eigen-equation. Equation (3) may then be written:

$$(4) \qquad A^{n} - T_1 A^{n-1} + T_2 A^{n-2} - T_3 A^{n-3} + \dots\dots + (-1)^{n} T_n = 0$$

If one multiplies (4) by A^{-1} and solves the resulting equation for A^{-1} one obtains:

$$(5) \qquad A^{-1} = -(-1)^{n} T_n (A^{n-1} - T_1 A^{n-2} + T_2 A^{n-3} - \dots T_{n-1} I) .$$

Granted that one is going to solve a system of equations by inversion of its matrix the ease with which one may find the inverse will depend somewhat on the ease with which one may compute the Ts . The statement about the elements in the Ts obeying the binomial law is somewhat empty as yet. We shall illustrate it with a matrix of the third order and by analogy write the general law. Given a matrix of the third order:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$T_0 = 1$ (always) of the empty order

$T_1 = \quad a_{11} + a_{22} + a_{33} = 3$ membere

$$T_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = 3 \text{ members}$$

$$T_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1 \text{ member}$$

Thus for a third order matrix we have the binomial law:

1 , 3 , 3 , 1

In a foubth order matrix we would have

1 , 4 , 6 , 4 , 1

one of the zero th order, four of the first order (the sum of the four diagonal terms), six of the second order, four of the third order, and one of the fourth order. For a fith order one would have

1 , 5 , 10 , 10 , 5 , 1

We shall illustrate this last theory by a numerical example.

Solve the following system by matrix inversion thru the eigen-value equation:

$$(6) \qquad \begin{vmatrix} 2 & 1 & 4 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -7 \\ 13 \\ -1 \end{vmatrix}$$

$$T_1 = 2 + 1 + 1 = 4$$

$$T_2 = 2 - 2 + 1 = 1$$

$$T_3 = -6$$

$$A = \begin{vmatrix} 2 & 1 & 4 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{vmatrix} \qquad A^2 = \begin{vmatrix} 8 & 3 & 8 \\ -4 & 1 & -8 \\ 3 & 1 & 5 \end{vmatrix}$$

$$4A = \begin{vmatrix} 8 & 4 & 16 \\ 0 & 4 & -16 \\ 4 & 0 & 4 \end{vmatrix} \qquad I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

According to (5) our inverse ought to be

$$(7) \qquad A^{-1} = (A^2 - 4A + I)/ 6$$

Putting in the values above one gets for the inverse

$$(8) \qquad A^{-1} = (1/6) \begin{vmatrix} 1 & -1 & -8 \\ -4 & -2 & 8 \\ -1 & 1 & 2 \end{vmatrix}$$

Multiplying both sides of (6) by (8) one obtains :

(9)
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{vmatrix} 2 \\ 1 \\ 3 \end{vmatrix}$$

(10)
$$(x_1 , x_2 , x_3) = (2 , 1 , 3)$$

Beckham Martin

Associate Professor of Physics
in Dayton University

Dayton, Ohio
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