Chap. 2

MUTATION GEOMETRY

Mutation Geometry is the science of intangible change. It is based on a single proposition of Mutation called omega ( ) and a single postulate called alpha ( ). This alpha postulate is implemented by a Mutation Diagram.

Before we state and prove this Proposition of Mutation, state and describe the alpha Postulate along with its implementing Mutation Diagram we should like to do some simple things Mutation-wise (not that the foundations of Mutation Geometry are not simple). They are very elementary notions.

1. Equation of a Straight Line Mutation-wise.

Find the equation of a straight thru two given points. Let a and b be the two vectors from an origin O to the two given points and r a vector from O to some point in the line passing thru the given points. See the sketch below:

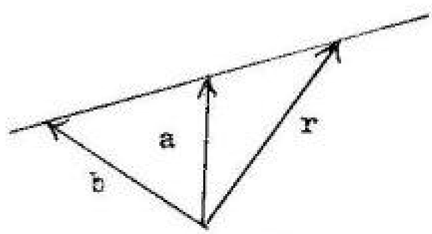

Fig. 12

$(r - b)$ is a segment of the line and $(a - b)^{\smile}$ is a segment perpendicular to the line. We may then write:

$$(1) \qquad (a - b)^{\smile} \cdot (r - b) = 0 :$$

$$(2) \qquad (a - b)^{\smile} \cdot r = \breve{a} \cdot b$$

Expression (2) is the Mutation equation of the required line. For example, if a and b are:

$$\begin{Bmatrix} 5, 1 \\ 2, 3 \end{Bmatrix}$$

we have $(3 - 1) x + (5 - 2) y = (3 - 1) 2 + (5 - 2) 3$

or $2x + 3y = 13$

One can do them mentally: (4, 1)

(2, 2)

$$x + 2y = 6$$

(7 , 2)

(4 , 6)

$$4x + 3y = 34$$

In the last example we subtract the 2 from the 6 to get the 4 for the coefficient of the x and we subtract the 4 from the 7 to get the 3 for the coeffivient of y. One then puts either of the given points into this and gets:

$$28 + 6 = 34 = 16 + 18 = 34$$

With a little practice one can just write down the answer at once. The equation of a plane thru three points may be easily written down or the equation of a circle thru three points. It is a matter of ease.

2. An Alpha Prototype Product

We call the scalar product a.b an alpha prototype product.

3. Omega Products

The double product (a . b) (c . d) is an omega type product.

4. Mechanical and Analytical Solutions of an Alpha type product in two Dimensions. Given:

(1) $a . r^1 = b$

For the mechanical solution for r^1 we put a circle on vector a as a diameter and with the origin O as center and radius b we cut this circle in two points A and B giving the two directions OA and OB for r^1. See the sketch below:

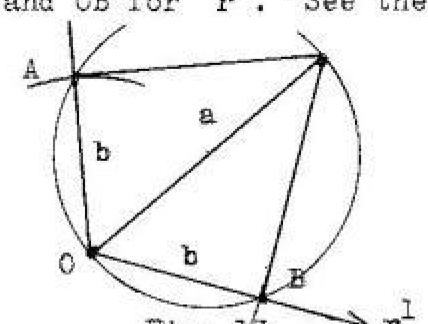

Fig. 13

This is so for the projection of a upon r^1 is b in both cases. The analutical solution is:

$$(2) \quad r^1 = (b\,a \ast \breve{a} (a^2 r^2 - b^2)^{\frac{1}{2}}) \,/\, a^2$$

If we put r^1 from (2) into (1) it satisfies and the square of both sides of (2) gives 1. Thus equation (2) is one solution. It is proven in the unpublished Mutation Geometry that (2) is the only solution of its type and thus one may use it with assurance of its correctness. There may be other solutions of a different type and with beautiful properties but if they exist we are not interested in them now. We have one solution (2) and we go with it. It is all we need now.

In linear programming we shall be interested primarily in polyhedrons, in particular hyperconvex polyhedrons formed by the hyper planes represented by the equations of constraint.

For the Mutation enlightenment of the students we turn aside, temporarily, to deal with apparent extraneous material and it may be for some who may not be able to see the connection now,

We shall not go too far in a wholy new science Mutation Geometry. We shall only prove the Proposition of Mutation, state the alpha Postulate, describe the Mutation Diagram and work a few problems so the student can see the Proposition in action. We shall also solve the primordial alpha prototype product splintered splintered from a number of omega products and assembled, at the choosing of the student, in accord with the alpha Postulate, into a single alpha product. The Mutation Diagram pinpoints each of the constituent parts in the final protype. After this we shall record (not prove here) some of the Monumental expressions From Mutation Geometry.

Mutation Geometry has generalized all the more important Propositions of college geometry such as the Simson Line Theorem, the Brocard Theorem, the Problem of Apollonius, etc. It can do every stunt of projective geometry, such as the construction of the points of intersection of a line with a conic given by five conditions when the conic is not drawn or any of the other marvelous doings of projective geometry without a great build-up. This is possible because Mutation Geometry has emancipated the human mind from the necessity of logical order and sequence. Who ever heard of any one writing a Conventional geometry without the theorems following eachother in a logical order. It is a necessity. Not so in Mutation Geometry. There is no sequence to a single thing: The Proposition of Mutation.

The essential operating principle of Mutation Geometry is in a Proposition of Mutation splintering omega type products into a sum of alpha type products which are grouped into a chosen single alpha type product called the primordial composite alpha prototype. It is generally identified with one of the splintered groups (the choice of which one being at the whim, fancy, or caprice of the student). These statements will not mean much till the student learns how to operate the New Science of Mutation Geometry.

5. The Proposition of Mutation

The single omega product a . r b.r can be splintered into two alpha type products:

$$(1) \quad a . r b . r = r^2 (a . b + a . \quad)/ 2$$

$$((2) \quad b \ r \ , \quad = b_0$$

Equation (1) is an algebraic statement of the Proposition of Mutation.

Equation (2) states that gamma is the symmetric of b with respect to r and in magnitude is equal to that of b.

Before we prove (1) we look a little further. The expression in (2) is a Mutation Diageam for a single omega product. Suppose we have a sum of omega products as:

$$(3) \quad S = a . r^1 b . r^1 + c . r^1 d . r^1 + e . r^1 f . r^1$$

and we splinter them with our Proposition of Mutation getting

$$(4) \quad S = (a . b + c . d + e . f)/2 + ((a . \alpha) + c . \gamma + e . \delta) /2$$

$$(5) \quad \begin{array}{cccc} \alpha & \wedge & b & r \\ \gamma & \wedge & d & r \\ \delta & \wedge & f & r \end{array}$$

Here (5) is our Mutation Diagram.

It is proven in the unpublished Mutation Geometry that the angle between alpha and gamma is the same as the angle between b and d, known vectors, and in the same way the angle between alpha and delta is the same as the angle between b and f, known vectors.

If this is so we may write, with the help of (5), equation (4) as:

$$(6) \quad S = M + N . \alpha$$

$$M = (a . b + c . d + e . f)/2$$

$$N = (b_0 \; a + d_0 \; \widehat{c} + f_0 \; \widehat{e})/2$$

Here $\widehat{c}$ and $\widehat{e}$ are called comigrates. $\widehat{c}$ is c rotated thru the angle b to d and in the direction b to d a known angle and direction according to the Mutation Diagram which is an implementation of the alpha Postulate. In the same way $\widehat{e}$ is e rotated thru the angle b to f and in the direction b to f a known angle and direction. We have chosen to group all the alpha products into the primordial alpha prototype product $N . \alpha$. One could have chosen to send them into any prototype as $L. \gamma$ if so desired. It is at the fancy of the operator. N is a known vector and can be drawn. If S should be known we could write equation (6) as:

$$(7) \qquad N . \alpha = P$$

$$P = S - M.$$

We can solve (7) either mechanically or analytically. See Fig. 13 and (2) following it.

If we know alpha (α) then the Mutation Diagram teaches us that r^1 is the bisector of the angle between the known vectors b and alpha (α). Thus we know r^1.

6. The Alpha Postulate:

The alpha and omega products are required to be tempo-locally invariant, time and local having nothing to do with their value. They may be mentally shifted (intangible change) tossed or herded from hither to yon without altering their value.

This Postulate is a cardinal principle of Mutation Geometry, The action in the Mutation Diagram is an implementation of the alpha Postulate.

7. Proof of the Proposition of Mutation.

We recopy here the algebraic expression to be proved:

$$(1) \quad a . r^r \; b . r^l = (a . b + a . \alpha) / 2$$

$$\alpha_\partial = b_0$$

$$\alpha \wedge b \quad r$$

See the sketch below.

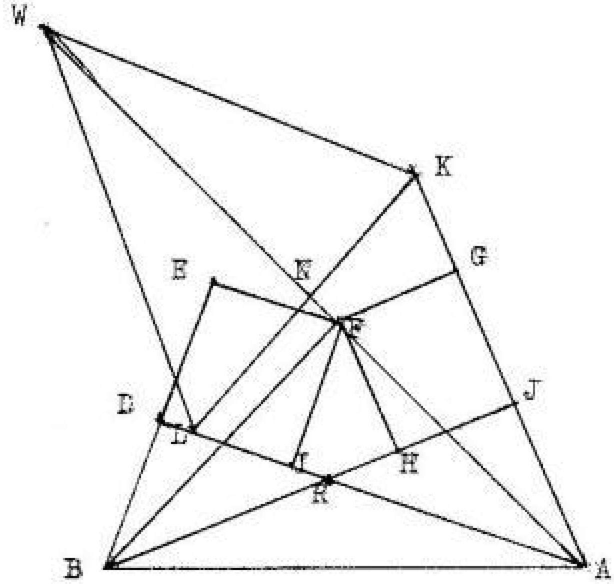

Fig. 14

Proposition of Mutation

Let AB represent a and AK b. Let the unit direction r[1] lie along AW. Draw a line AL equal to AK and making angle KAW equal to angle WAL. Complete the rhombus on AK and AL as KALW. Draw BD, BF, and BJ perpendicular to AL, AW, and AK respectively. Draw FE, FI, FH, and FG perpendicular to BD, AL, BJ, and AK respectively. N is the common point to lines AW and KL. M is the common point to lines AW and BJ. It will now be shown that BF is the bisector of angle EBH.

In right triangles BFM and AMJ there is a common angle at M so angle FBM and Maj are equal. In right triangles BDR and AFR there is a common angle at R so angle RBD and RAJ are equal. Now angle RAF and MAJ are equal by construction. Thus angle RBD and FBM are equal. The followinf relations may be written:

(1) JG = HF = FE = DI

(2) FI = FG

(3) AG = AI = AJ + JG = AJ + DI = AJ + AD - AI

= AJ + AD - AG. Thus

(4) 2 AG = AJ + AD.

Multiply both sides of (4) by AK and get:

(5) 2 AK AG = AK AJ + AK AD.

From the similar right triangles AFG and ANK one has:

(6) AK / AF = AN / AG or

(7) AK AG = AF AN.

Put (7) into (5) and get:

(8) 2 AF AN = AK AJ + AK AD.

(9) $AF = a . r^1$

(10) $AN = b . r^1$

(11) AK AJ = a . b

(12) AK AD = AL AD = a .

Here AL equal to AK equal to b_0 in magnitude , is represented by alpha ( ).

Put (9), (10), (11) and (12) into (8) and get:

(13) $a . r^1 b . r^1 = (a . b + a .) /2$

which is the Proposition of Mutation. The truth of this may also be demonstrated synthetically in the folloeing way:

(14) $a . r^1 b . r^1 = a . \frac{(b +)}{(b +)_0} b . \frac{(b +)}{(b +)_0}$

$= (a . b + a .) (b^2 + b .)/ (b +)^2$

$(b +)^2 = b^2 + 2 b . + b^2 = 2 (b^2 + b .)$

Put this last value back into (1 4) and get

(1 4) $a . r^1 b . r^1 = (a . b + a .)/ 2 .$

To a physicist the geometric visual demonstration of this Proposition of Mutation would seem the more pleasing. The synthetic proof lends credence to the truth of the Mutation.

The demonstration of the truth of that proposition is a great stride in the organization of a truly Pan-Geometry which has unified the field of geometry.

In projective geometry one meets the beautiful theorems of Desargues, Pascal, Brianchon, Poncelet, and host of others. They require a lot of build up with their attendant requirements of logical sequence and order. Who ever heard of any one writing a conventional geometry without arranging the theorems in some sort of logical order and sequence ?

Mutation Geometry has emancipated the human mind from the bondage of logical order and sequence. There is no sequence to one (the Proposition of Mutation) , the single operating princible for the field of geometry.

There is another pleasing aspect of Mutation Geometry : It sets Linear Programming on an entirely new and satisfactory foundation.

Before we return to the main purpose of this book, linear programming, we do a couple of elementary problems of college geometry just to watch the alpha postulate and proposition of mutation in action.

If the student of college geometry should be required to make a certain construction with ruler and compass he generally makes a drawing as near like the required drawing as possible and he starts out to find some relations between its parts that will enable him to actually make the construction. Some times he is successful but often no relations are to be found and he is stymied and is reduced to sitting and looking and if no success he tries looking and sitting both wasteful occupations.

Mutation Geometry attacks its problems head on and does not have to to find relations or connections with previous problems. It moves smoothly to the attack. It has only one thing to do. It does not take long to see the advantages of such a geometry over the older conventional geometries.

Problem. Thru a given point P on the circumference of a given circle whose center is O construct two chords making a given angle K with each other so that their sum shall equal to a given line segment S.

Let a be the diameter,of the given circle, thru point P and b and c the unit vectors along the required chords. The lengths of the required chords are a . b and a .c . One then may write:

$$(15) \qquad a \cdot b + a \cdot c = S = (a + \hat{a}) \cdot b = S.$$
$$= d \cdot b = S$$

where $d = a + \hat{a}$. and $\hat{a}$ is the migrate of a.

Mechanically we put a circle on the known line d as a diameter and with P as a center and a radius S cut this circle in points A and B giving the two directions P A and PB for b and c. One then draws lines thru point P making angle K with PA and PB giving the other chord. One will have two, one or no solutions according as d_0 is greater than, equal to or less than S. One could just as easily have solved the problem where there were n chords making given angles with each other. In that case:

$$(16) \quad d \cdot b = S$$

$$d = \hat{a}_1 + \hat{a}_2 + \hat{a}_3 \ldots \hat{a}_n$$

The comigrates a_i are known constructible vectors and one has the same solution technique as in the case of two chords.

All constructible problems of the geometric world are either in the alpha or omega category. We now do a problem that is in the omega category:

Thru a given point on the circumference of a given circle draw two chords making a given angle with each other so that the product of the required chords shall equal the square on a given line segment. Let P be the given point and K the given angle and S the given line segment. One then can write:

$$(17) \quad a \cdot b \; a \cdot c = S^2 = a \cdot b \; \hat{a} \cdot b = S^2.$$

Splintering (17) with the omega proposition we get:

$$(18) \quad a \cdot \hat{a} + a_0 \hat{a} \cdot e = 2 S^2 \quad \text{whence}$$

$$(19) \quad \hat{a} \cdot e = f = (2 S^2 - a \cdot \hat{a}) / a_0$$

$$(20) \quad e \wedge a \quad b$$

To solve equation (19) we put a circle on $\hat{a}$ as a diameter and with P as a center and f as a radius cut this circle in the two points A and B giving the two directions P A and P B for the e. With e known we get b from Equation (20) as the bisector of the angle between a and e. To find the other chord one draws a line thru P making angle K with the b direction. There will be two, one or no solution as a is greater than, equal to, or less than f. Let us now complicate the problem a bit and write:

Thru a given point P on the circumference of a given circle construct two chords making a given angle K with each other such that the sum of the squares on the two chords plus their product shall equal the square on a given line segment S. We may now write:

$$(21) \quad (a \cdot b)^2 + (a \cdot c)^2 + (a \cdot b) (a \cdot c) = S^2$$

where b and c are the unit directions along the required chords. In accordance with the alpha postulate the last equation may be written:

$$(22) \quad (a \cdot b)^2 + (\frown{a} \cdot b)^2 + (a \cdot b)(\frown{a} \cdot b) = S^2$$

Splintering equation (2 2) we get:

$$(23) \quad (a^2 + a_0 a \cdot d) + (a^2 + a_0 \frown{a} \cdot e) + (a \cdot \frown{a} + a_0 \frown{a} \cdot d) = 2 S^2 .$$

$$(24) \quad d \wedge a \quad b$$
$$e \wedge a \quad b$$

Grouping in (2 3) we get

$$(25) \quad M \cdot d = N .$$

$$M = a + \frown{a} + \frown{A}$$

$$N = (2 S^2 - 2 a^2 - a \cdot \frown{a}) / a_0 .$$

Equation (2 5) is an alpha prototype whose solution is standard: Put a circle on M as a diameter and with P as a center and radius N cut this circle in the two points A and B giving the required chords when produced. There will be two, one, or no solutions according as M is greater than, equal to, or less than N.

We state again that the problems of the geometric world are in either the alpha or omega category.

The Proposition of Mutation splinters those in the omega category into a sum of alpha products and the Mutation Diagram sweeps them into a chosen composite primordial alpha prototype product which we have shown how to solve both mechanically and analytically. Thus one can, at least in principle, solve the problems in the geometric world in a straight forward manner nor is one bound by any law of order and sequence.

Mutation Geometry has generalized all the worth-while theorems of college geometry such as the problem of Apollonius, the Simson line theorem, the Brocard theorem, etc. The Generalization of the problem of Apollonius is to construct a circle cutting three given circles A, B, and C at given angles alpha (α), beta (β), and gamma (γ) respectively where the angles are unrestricted. There is only one circle that will do this. In the original problem of Apollonius he studied the construction of a circle that would be tangent to three given circles and he gave the correct solution: 8 possible circles. There are two ways that a circle may be tangent to a given circle. It may touch either externally or internally and theus for three circles one gets $2^3 = 8$. Now if we specify that the required circle is to be tangent to circle A externally (cut at 0 degrees) and pass around circle B (cut at 180 degrees) and touch circle C externally (cut at 0 degrees) then there is only one circle possible. The word tangency has a double meaning.

There are people in the geometric world who are still looking for an 8 th degree equation that would give the 8 radii of the Apollonian problem.

Mutation Geometry says there is no such nor is one to be expected. It is of the second degree where one of the roots is always negative leaving only one circle possible. To look for an 8 th degree equation is an idle dream and an exercise in futility.

We have not taken time here to make the actual drawings for the problems worked nor that for the Apollonian generalization . The actual drawing for the generalization of the Apollonian Problem is a beautiful configuration and if he could see it I believe he would evince some surprise and satisfaction at Mutation Geometry,s power over his pet problem.

Those who are primarily interested in linear programming will perhaps chafe at this aversion into an apparent unrelated field but one often needs a background in the workings of the New Science of Mutation Geometry and this aversion should enhance one,s grasp of linear programming which Mutation-wise has to do with the geometry of hyper convex polyhedrons. A smoothe working knowledge of Mutation Geometry will prove to be a powerful tool in the solution of linear programming problems.

It is a temptation to merely record here some of the monuments of Mutation Geometry in the field of Cartesian analytics but we shall not push the patience of the reader too far and just refer him to a presentation made before the Ohio Section of the American Mathematical Assn. meeting at Miami University in Oxford, Ohion. See the Aug-Sept issue of the American Mathematical Monthly page 645, 1959.

8. Solution of a System of n Linear Equations in n Variables.

Consider the system:

$$(1)\quad \begin{array}{c} A_1 \cdot r = b_1 \\ \cdots\cdots\cdots\cdots \\ A_n \cdot r = b_n \end{array}$$

$$r = (x_1 , x_2 , \ldots x_n)$$

$$A_j = (a_{j1} , A_{j2} , \ldots A_{jn}) , (j = 1, 2, \ldots n)$$

A formal solution of (1) is :

$$(2) \qquad r = b_1 A_1^{-1} + b_2 A_2^{-1} + \ldots b_n A_n^{-1} .$$

where A_j^{-1} is the reciprocal vector to A_j .

$$A_i \cdot A_j^{-1} = 1, \quad i = j$$
$$= 0, \quad i \neq j$$

System (1) may be put into the form:

$$(3) \qquad \begin{array}{l} A_1 \cdot r = b_1 \\ B_2 \cdot r = 0 \\ \ldots\ldots\ldots\ldots \\ B_n \cdot r = 0. \end{array}$$

where $B_j = (b_j A_1 - b_1 A_j)$

A formal solution of (3) is the single term:

$$(4) \qquad r = b_1 \cdot A_1^{-1} .$$

The A_1^{-1} in (4) is not the same as the A_1^{-1} in (2) .

As an illustration of (4) we shall do a couple of simple examples.

9. Gamma vectors

Before we do the illustrative examples we should like to describe the construction of gamma vectors which will play a dominant role in the Mutation Geometry theory of linear programming.

If we have n vectors a, b, c, d .... with n components each and we omit one of them, say a , we write :

$$\gamma^{-a}$$

for the vector that is normal to each of the remaining n - 1 vectors b, c, d, ... When we are dealing with the maximization of some linear objective function the product

$$a \cdot \gamma^{-a}$$

is always to be negative and positive when dealing with minimization problems. We shall show how to construct the gamma vectors in a number of ways. One sees that there are n gamma vectors for n vectors. One may also write:

$$(1) \qquad a^{-1} = \gamma^{-a} / a \cdot \gamma^{-a}$$

as the reciprocal of vector a, for

$$a \cdot a^{-1} = (a \cdot \gamma^{-a}) / (a \cdot \gamma^{-a}) = 1$$

and it is so constructed that it is perpendicular to the other n - 1 vectors. We have devises several ways of constructing the gamma vectors for they are at the heart of linear programming in the styling of Mutation Geometry.

We shall now do the illustrative examples promised. Solve the system:

$$(2) \qquad \begin{vmatrix} 1 & +2 & -1 \\ 2 & +1 & -1 \\ -1 & -1 & +2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}$$

This may be written in the form:

$$(3) \qquad \begin{matrix} a \\ b \\ c \end{matrix} \begin{vmatrix} 1 & +2 & -1 \\ 0 & +3 & -1 \\ 4 & +7 & -5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

To calculate γ^{-a} we take column cofactors of vectors b and c in order from left to right getting:

$$\gamma^{-a} = -8 - 4 - 12$$

$$a \cdot \gamma^{-a} = -8 - 8 + 12 = -4$$

$$r = b_1 A^{-1} = 1 (-8 - 4 - 12)/-4 = 2 + 1 + 3$$

We usually omit the unit hyper-vectors as they cause no confusion and the notation is much smoother without them. We have gone to a bit of detail with this simple example so that the reader can see the process without any confusing element.

Example 2. Solve the system:

$$\begin{vmatrix} 1 & -2 & +1 & +1 \\ 2 & +1 & -1 & -1 \\ -1 & +0 & +2 & +0 \\ -2 & +3 & -1 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} \begin{matrix} = \\ = \\ = \\ = \end{matrix} \begin{vmatrix} 2 \\ 3 \\ -4 \\ -3 \end{vmatrix}$$

which may be written:

$$\begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{vmatrix} 1 & -2 & +1 & +1 \\ -1 & -8 & +5 & +5 \\ 1 & -4 & +4 & +2 \\ -1 & +0 & +1 & +1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Again we take column cofactors of rows b, c, and d in order from left to right getting:

$$\gamma^{-a} = 2 + 1 - 1 + 3$$

where we have canceled a common factor -8 from the cofactors.

$$a \cdot \gamma^{-a} = 2$$

$$r = b_1 A^{-1} = 2 (2 + 1 - 1 + 3) / 2 .$$

$$= (2 + 1 - 1 + 3 .$$

We know also that :

$$(4) \quad b \cdot \gamma^{-a} = c \cdot \gamma^{-a} = d \cdot \gamma^{-a} = 0$$

since each expression represents a determinant with two of its rows equal thus, γ^{-a} is perpendicular to each of the vectors b, c, and d. A like relation holds for the other gamma vectors. The gamma vectors are of prime importance in linear programming in the style of Mutation Geometry.

10. Polarization in a Plane and in Space.

In linear programming a large number of the relations are inequalities. They are governed by the two signs: $<$ and $>$. The sign $>$ means greater than and the sign $<$ means less than.

Consider the equation:

$$(1) \quad 2 x + 3 y = 7.$$

Two points on this line are (2, 1) and (5,-1). When we put these points into the line they satisfy:

$$2 (2) + 3 (1) = 4 + 3 = 7$$

$$2 (5) + 3 (-1) = 10 - 3 = 7$$

When the point (1, 1), which is on the near side of the line to the origin, is put into the equation we get:

$$2(1) + 3(1) = 2 + 3 = 5 < 7.$$

In the same way all points (x, y) on the near side of the line to the origin give:

$$2x + 3y < 7.$$

This inequality is a criterion for points on the near side of the line to the origin.

When we put the point (2, 2), which is on the far side of the line from the origin, we get:

$$2(2) + 3(2) = 4 + 6 = 10 > 7.$$

and in like manner for all points (x, y) which are on the far side of the line from the origin:

$$2x + 3y > 7.$$

We choose an arbitrary point (1, 2) and want to know whether it is on the near or far side of the line to the origin. We put the point into the line equation and get:

$$2(1) + 3(2) = 2 + 6 = 8 > 7$$

and it answers: on the far side. The line is said to polarize the plane into regions $<$ $=$ or $>$, according as the points are on its near side, on, or on its far side from the origin.

There is no need for slack variables in the new formulation of linear programming. The notions of polarization will play a significant role in the new styling.

In the same way a hyper-plane of n-1 dimensions is said to polarize a hyper-space of n dimensions.

$$a \cdot r < = > b$$

according as the point is on the near side, on, or on the far side of the plane from the origin .

1 1. Polarization and Polyhedrons.

The polyhedrons with which we shall deal in linear programming are convex (every line joining any two points in the polyhedron lies wholy in the polyhedron). The polyhedrons may be either closed or open.

1 2. Vertices of polyhedrons.

If a polyhedron is represented by m relations of inequality as:

$$
(1) \qquad
\begin{array}{l}
A_1 \cdot r \leq b_1 \\
A_2 \cdot r \leq b_2 \\
\cdots\cdots\cdots\cdots \\
A_n \cdot r \leq b_n \\
\cdots\cdots\cdots\cdots \\
A_m \cdot r \leq b_m
\end{array}
$$

$$m > n \quad (\text{ always })$$

then the vertices of the representative polyhedron are each the intersection of n of the m equations which vertices each satisfies the sustem of inequalities in (1). n represents the dimensionality of the space under consideration.

Consider the two dimensional polygon represented by the set of inequalities:

(1)	$-1\,x_1 + 0\,x_2$	≤ 0
(2)	$0\,x_1 - 1\,x_2$	≤ 0
(3)	$1\,x_1 - 2\,x_2$	≤ 3
(4)	$1\,x_1 + 1\,x_2$	≤ 9
(5)	$-3\,x_1 + 1\,x_2$	≤ 1
(6)	$1\,x_1 + 2\,x_2$	≤ 14
(7)	$2\,x_1 - 1\,x_2$	≤ 9
(8)	$-2\,x_1 + 1\,x_2$	≤ 2

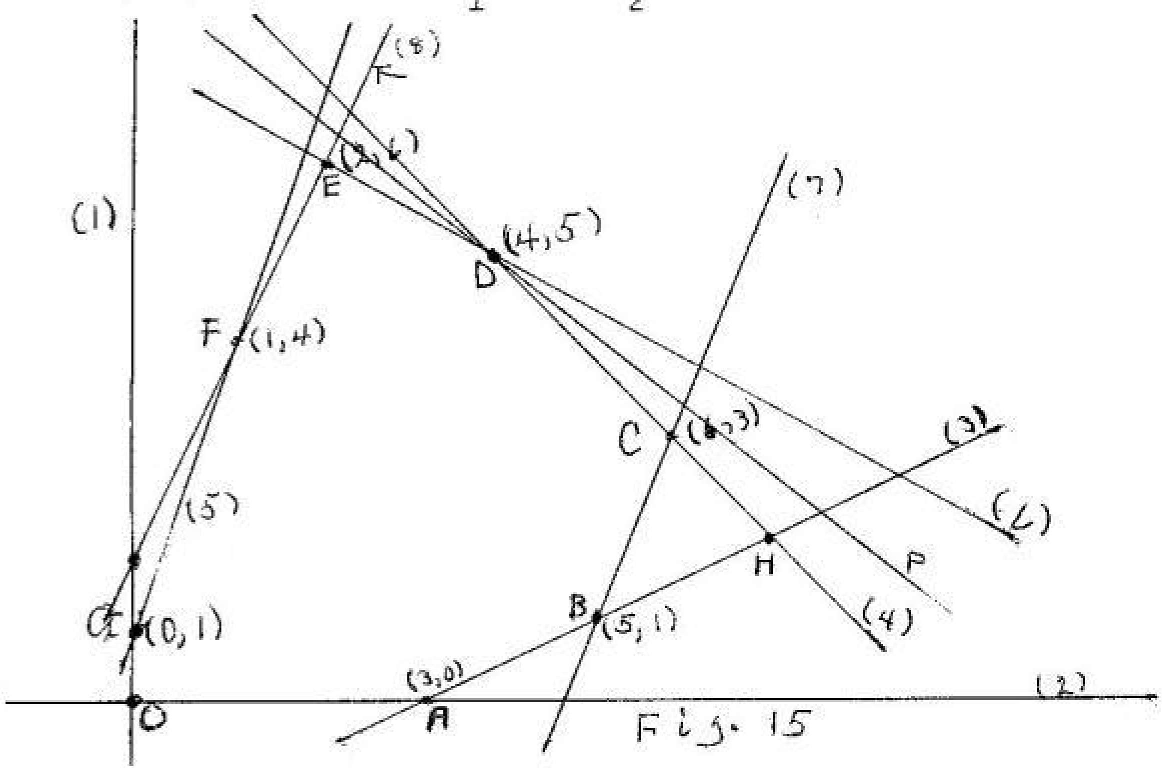

Fig. 15

The vertices are: O (0, 0), A (3, 0), B (5, 1) C (6, 3), D (4, 5), E (2, 6), F (1, 4), G (0, 1).

Point H (7, 2), is the intersection of lines (3) and (4) but it is not a vertex since it does not satisfy relation (7). As we shall continue to see the notion of polarization is a simplifying element in the whole field of linear programming.

The interior region of the polyhedron of constraint is said to be a feasibility region and every point in the feasibility region satisfies the system of constraint but only t those points in the feasibility region which are the intersection of n of the inequations of constraint, taken as equations, are verfices. Finding the vertices is of prime importance. The gamma vectors will play their full part in this. Later we shall show how to generate the vertices of a convex polyhedron in n dimensions.

Before we do that we turn aside, temporarily, to deal with eigen-vectors and eigen-values.

13. Eigen-vectors and Eigen-values.

Let A be a square matrix of n rows and columns and r a vector with n componets:

(1) $r = x_1 + x_2 + x_3 + \dots x_n$

where for simplicity we have omitted the unit hyper vectors i_j.

When A is applied to just any vector p one generally gets another vector s, say, which has a different magnitude and direction than that of p:

(2) $A \cdot p = s.$

We want to find a vector r so that A will not change its direction but at most change its size. In this case we write:

(3) $A \cdot r = k r.$

Here r is called an eigen-vector of the matrix A and the scalar multiplier k is called an eigen-value of the matrix A.

One may write (3) in the form:

(4) $(A - k I) \cdot r = 0.$

Equation (4) has non-trivial solutions for r only when:

(5) $(A - k I)_0 = 0.$

Expanding equation (5) one gets:

$$(6) \quad k^n - T_1 K^{n-1} + T_2 k^{n-2} - T_3 k^{n-3} + .. (- 1)^n T_n = 0.$$

where T_j is the sum of the determinants of order j down the main diagonal.

The number of these determinants obeys the binomial law according to Mutation Geometry. This knowledge saves the messy expansion of large determinants. For example, for a third o order determinant we would have:

T_0	T_1	T_2	T_3
1	3	3	1

which means one would have one determinant of the 0 th order, 3 of the first order, 3 of the second order, and 1 of the 3rd order. For a 4th order determinant we would have:

T_0	T_1	T_2	T_3	T_4
1	4	6	4	1

and in general the binomial law.

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$T_0 = 1$$

$$T_1 = a_{11} + a_{22} + a_{33}$$

$$T_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$T_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We do a numerical example of the third order, calculating the three eigen values and their corresponding eigen-vectors.

Numerical example: Given the matrix:

$$A = \begin{matrix} 2 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & -2 & 4 \end{matrix}$$

$$T_1 = 2 + 3 + 4 = 9$$

$$T_2 = \begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix} + \begin{matrix} 3 & -1 \\ -2 & 4 \end{matrix} + \begin{matrix} 2 & -1 \\ 3 & 4 \end{matrix} = 26$$

$$T_3 = 24.$$

$$k^3 - 9k^2 + 26k - 24 = 0.$$

$$(k - 2)(k - 3)(k - 4) = 0.$$

$$k_1 = 2, \quad k_2 = 3, \quad k_3 = 4.$$

To find the eigen-vector r_1 corresponding to k_1 we put k_1 into (4) and get for the first two rows:

$$\begin{matrix} 0 & 1 & -1 \\ 1 & 1 & -1 \end{matrix}$$

taking a gamma vector of this (column cofactors) we get:

$$r_1 = 0 \quad -1 \quad -1$$

In the same way we get for k_2 :

$$\begin{matrix} -1 & 1 & 1 \\ 1 & 0 & -1 \end{matrix}$$

and the gamma for this is:

$$r_2 = -1 \quad +0 \quad -1$$

For k_3:

$$\begin{matrix} -2 & +1 & -1 \\ 1 & -1 & -1 \end{matrix}$$

The gamma for this is:

$$r_3 = -2 \quad -3 \quad +1 .$$

$$A = \begin{matrix} 2 & +1 & -1 \\ 1 & +3 & -1 \\ 3 & -2 & +4 \end{matrix}$$

$$\begin{matrix} k_1 = 2, & r_1 = 0 - 1 - 1 \\ k_2 = 3, & r_2 = 1 + 2 + 1 \\ k_3 = 4, & r_3 = -2 - 3 + 1 \end{matrix}$$

In this demonstration we have calculated the eigen-values first then used them to easily get the corresponding eigen-vectors. Now suppose want to calculate the eigen-vectors first and then use them to get the eigen-values. We may write our eigen equation as

$$(\ 7\ ) \qquad A \cdot r^1 = k\, r^1$$

where we have cancelled the magnitude r_0 from each side of the eigen-equation. Multiply both sides of the eigen-equation (7) by r^1 and get:

$$(\ 8\ ) \qquad r^1 \cdot A \cdot r^1 = k\, r^1 \cdot r^1 = k$$

If we can find r^1 then equation (8) will give k, the eigen value. Multiply equation (7) by $\breve{r}$ and get:

$$(\ 9\ ) \qquad \breve{r} \cdot A \cdot r = 0$$

From Mutation Geometry we have the identities:

$$(\ 10\ ) \qquad r_3 = (\ i_1 \cdot r\ )\ i_2 - (\ i_2 \cdot r\ )\ i_1$$

$$(\ 11\ ) \qquad r_2 = (\ i_1 \cdot r\ )\ i_3 - (\ i_3 \cdot r\ )\ i_1$$

$$(\ 12\ ) \qquad r_1 = (\ i_2 \cdot r\ )\ i_3 - (\ i_3 \cdot r\ )\ i_2$$

Multiply equation (7) by equations (10) , (11) and (12) and get:

$$(\ 13\ ) \quad r \cdot (\ i_1\ (\ a_{21} i_1 + a_{22}\, i_2 + a_{23}\, i_3\ ) - i_2 (\ a_{11}\, i_1 + a_{12}\, i_2 + a_{13}\, i_3\ )\ ) \cdot r = 0$$

$$((\ 14\ ) \quad r \cdot (\ i_1\ (\ a_{31} i_1 + a_{32}\, i_2 + a_{33}\, i_3) - i_3 (\ a_{11} i_1 + a_{12} i_2 + a_{13} i_3)) \cdot r = 0$$

$$(15) \quad r . (i_2 (a_{31} i_1 + a_{32} i_2 + a_{33} i_3) - i_3 (a_{21} i_1 + a_{22} i_2 + a_{23} i_3)) . r = 0.$$

One of these is redundant, say (15), for it can be obtained from the other two. We write our eigen-vector as:

$$(16) \quad r = x_1 (i_1 + h_2 i_2 + h_3 i_3)$$

Put equation (16) into (13) and (14) and get:

$$(17) \quad a_{21} + a_{22} h_2 + a_{23} h_3 = h_2 (a_{11} + a_{12} h_2 + a_{13} h_3)$$

$$(18) \quad a_{31} + a_{32} h_2 + a_{33} h_3 = h_3 (a_{11} + a_{12} h_2 + a_{13} h_3)$$

We thus arrive at two quadratics in h_2 and h_3. For our matrix:

$$A = \begin{matrix} 2 & + & 1 & - & 1 \\ 1 & + & 3 & - & 1 \\ 3 & - & 2 & + & 4 \end{matrix}$$

the two quadratics become:

$$(19) \quad 1 + 3 h_2 - h_3 = h_2 (2 + h_2 - h_3)$$

$$(20) \quad 3 - 2 h_2 + 4 h_3 = h_3 (2 + h_2 - h_3).$$

By inspection , in this simple case, one set of values is:

$$h_2 = 2$$

$$h_3 = 1$$

then

$$r^1 = (i_1 + 2 i_2 + i_3) / \quad 6$$

$$k = r^1 . A . r^1 = 3$$

which is one of the eigen-values already obtained by the first method. Another set of values of (19) and (20) is:

$$h_2 = 3/2$$

$$h_3 = -1/2$$

$$r = (i_1 + 3/2 i_2 - 1/2 i_3) = (2 + 3 - 1)$$

We shall find the corresponding k in a slightly different way just fot variety. We write:

$A \cdot r = k r$

$A \cdot r = 8 + 12 - 4 = 4 (2 + 3 - 1) = 4 r = k r$

so $k = 4$.

When x_1 is 0 our eigen-vector cannot be written :

$r = x_1 (1 + h_2 + h_3)$. Instead one may write:

(21) $r = x_2 (h_1 + 1 + h_3)$ or

(22) $r = x_3 (h_1 + h_2 + 1)$.

Using equatiom (23) one finds, as one set:

$h_1 = 0$

$h_2 = 1$

$r = 0 + 1 + 1$

$A \cdot r = 0 + 2 + 2 = 2 (0 + 1 + 1) = 2 r$

So $k = 2$. The three eigen-values corresponding to our three eigen-vectors are:

$r_1 = 0 + 1 + 1.$ $k_1 = 2$

$r_2 = 1 + 2 + 1,$ $K_2 = 3$

$r_3 = 2 + 3 - 1,$ $k_3 = 4$

which agree with the previous calculation. We now write the generalization for n dimensions:

$$A = \begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix}$$

$r = x_1 (1 + h_2 + h_3 + ;\ldots h_n)$.

$$a_{21} + a_{22} h_2 + .. a_{2n} h_n = h_2(a_{11} + a_{12}h_2 + .. a_{1n}h_n)$$

$$(23) \quad a_{31} + a_{32} h_2 + .. a_{3n} h_n = h_3(\dots\dots\dots\dots)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1} + a_{n2} h_2 + .. a_{nn} h_n = h_n(a_{11} + a_{12}h_2 + .. a_{1n}h_n))$$

The system of quadratic equations (23) is a pioneering one from the New Science of Mutation Geometry.

When the system is large these quadratics lend themselves easily to numerical solutions.

We shall call this scheme of calculating eigen-vectors and eigen-values the H - Way, and that in (6) the T - Way.

For large systems the H - Way seems the maore tractible. For small systems there does not seem much difference. Experience will help one to decide,in either case.