Chapter 4.

1. Linear Programming.

We shall first consider the maximization problem. We may write: maximize the linear objective function

$$(1) \qquad P = c \cdot r$$

subject to the system of constraints:

$$(2) \qquad A \cdot r \leq b.$$

Here c and r are n dimensional row vectors

$$c = c_1 + c_2 + \dots c_n$$

$$r = x_1 + x_2 + \dots x_n.$$

b is an m dimensional column vector

$$b = b_1 + b_2 + \dots\dots b_m$$

where $m > n$ always.

A is an m by n matrix.

The system of inequalities in (2) represents a polyhedron and the solution of n of the m inequalities, taken as equations, properly chosen, represents a point (a vertex of the polyhedron).

The vector to this vertex we call R and we associate an index with this as: R (( ), ( ), ( ), ... (n)) where the n parentheses contain the numbers of the equations in A whose solution give R.

We shall always indicate the normals of the equations whose numbers are in the index by (a, b, c, d, ...). If we omit one of the equations, say a, from the index we can calculate a γ for the remaining n - 1 vectors, the normals with n components. We denote that gamma by :

$$\gamma^{-a}$$

and in a similar way we get γ^{-b}, γ^{-c}, γ^{-d} ....

We, for the most part, calculate only n - 1 of the gamma,s for any one index because we do not need or already have the gamma on which we arrived at the point (vertex) R. The various gamma of any one index give the directions of the neighbors of the vertex whose vector is R. We may call the first point (vertex) with which we start R_0 . The neighbors then may be written:

$$(3) \qquad R_n = R_0 + t_n \gamma^{-n}$$

where t_n is a scalar to be determined in order that the point R_n^n be a vertex of the polyhedron of constraint.

To find the various t values in (3) we put (3) into every equation of the constraint system whose number is not in the index of R_0 and take the smallest + value of t found for each one. These values of t, when put back into (3), give points (vertices) on the polyhedron of constraint. The constraint system is always written so that the column vector b has all its components positive, then the normals of the hyper-planes in A will point from the origin to the plane.

Suppose we are leaving a vertex of the polyhedron located on a plane whose normal is a then the γ leaving this vertex on the plane whose normal is a for a neighboring vertex must leave that plane on the side next to the origin in order for the point reached to satisfy the constraint system. This has to be observed. This is in accord with the notion of Polarization . What is the same thing:

$$a \cdot \gamma^{-a}$$

must have a negative value. The sign of the gamma may always be adjusted so that the above expression has a negative value. That it should have a negative value can be seen from the sketch below.

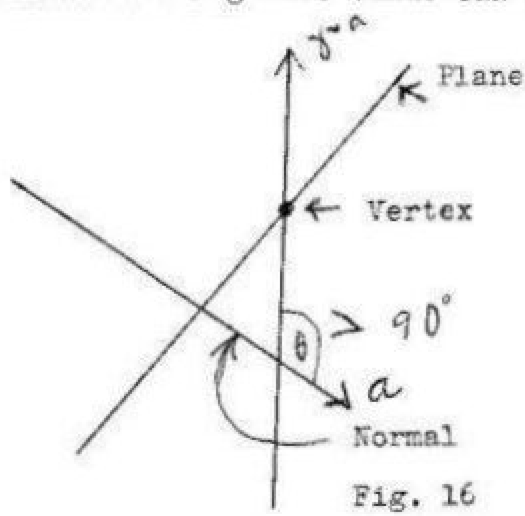

Fig. 16

If we are dealing with minimization problems then our constraint system is written:

$$A \cdot r \geq b$$

then $$a \cdot \gamma^{-a}$$

must have a positive sign according to the notion of polarization in order that the vertex attained will satisfy the constraint system. The departing γ must leave the plane on the side of the plane away from the origin. This can be seen from the sketch below.

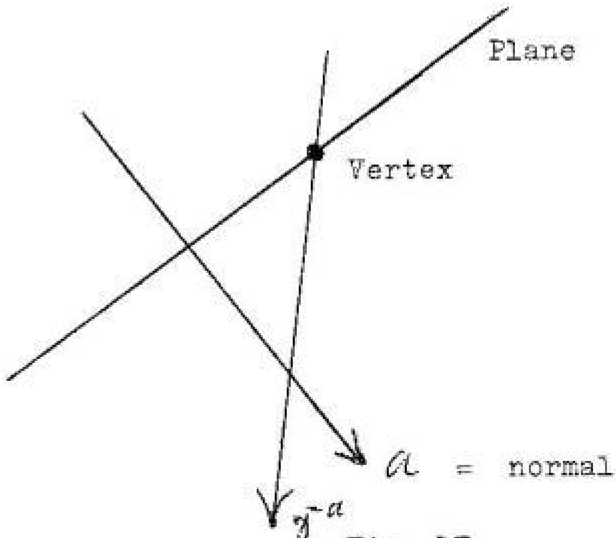

Fig. 17

If now one can find an initial vertex (and we can do it) of the polyhedron of constraint then our γ vectors will enable us to sample the neighbors of any vertex and choose the maximum of them or minimum of them and to repeat the proces till one finds the desired max. or min. as desired.

Note that here we have no use for: slack variables, expansion bases, degeneracy, or cycling. Mutation Geometr y has disposed of all such baggage and put linear programming on a more satisfactory foundation.

Having found an initial vertex whose vector is R_0 we put this R_0 into the objective function and get:

$$P_0 = c \cdot R_0$$

To find the value of the objective function for a neighboring vertex we write:

$$(4) \qquad R_j = R_0 + t_j \gamma_J$$

$$P_j = c \cdot R_j = c \cdot (R_0 + t_j \gamma_J)$$

$$= c \cdot R_0 + t_j c \cdot \gamma_J$$

$$(5) \qquad P_j = P_0 + t_j c \cdot \gamma_J$$

Each t_j is always positive and we want P_j to be equal tp or greater than P_0 when dealing with maximum problems and so in that case we use only those γ which give a positive value with the cost vector c. For Min . problems the opposite condition holds. This fact saves a lot of usless computation.

In maximization problems any vertex all of whose neighbors give negative values of c . gives the maximum value of the objective function P.

In minimization problems any vertex all of whose neighbors give positive values for c . gives the minimum value of the objective function P.

Consider, for a moment, a simple closed polygon and the line representing the objective function. The normal to this line is the vector c, aconstant in magnitude and direction. See Fig. 18. Let the objective function line whose normal is c take various positions across the polygon of constraint as $M_1 N_1$, $M_2 N_2$, $M_3 N_3$ till it reaches some point as C where M and N coincide. The value of the objective function P for every point on segment M N increases as M N recedes from the origin according to polarization and these points satisfy the system of constraints for maximum problems, for they are on the near side of each line of the constraint system to the origin. In the limit, as M N recedes from the origin, there is only one point on M N which satisfies the constraint system, namely some point as C. This point is also the intersection of the two lines B C and C D and thus by definition is a vertex of the polyhedron of constraint. Also the objective line thru C being the farthest from the origin takes on its maximum value according to polarization.

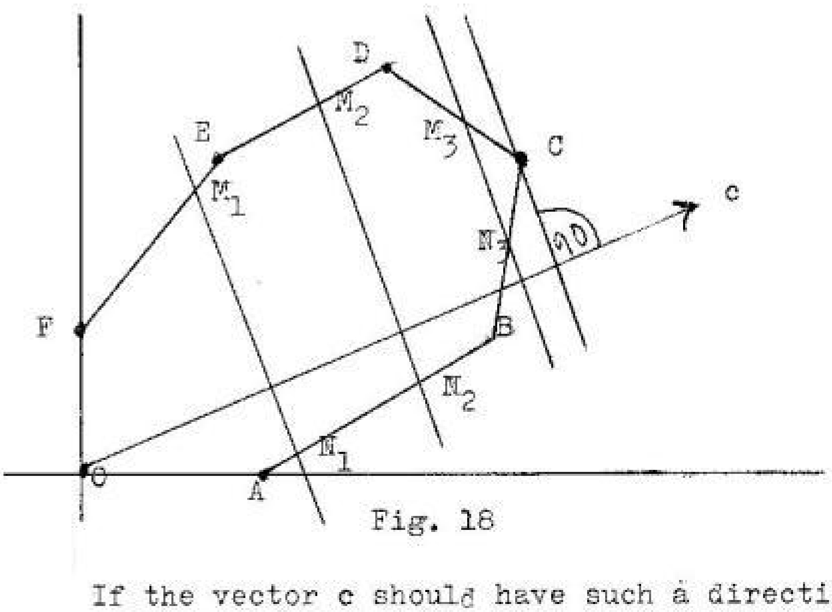

Fig. 18

If the vector c should have such a direction that the line MN in the limit should coincide with a chord of the polygon of constraint such as BC then every point on BC would give the same maximum value of the objective function as points Band C.

The same principle holds for a polyhedron in n dimensions. The plane of the objective function, in the limit, reaches a point (vertex) of the polyhedron or it coincides with one of the faces of the polyhedron and all the vertices on that face give the same maximum of the objective function. The same principle can be taken for minimum problems.

The principle of polarization enables one to reach conclusions that are belabored in long " proofs " in many texts in conventional linear programming.

2. Initial Vertices.

We start first with the maximization of the objective function P. We divide each term in each row of the matrix A into the b component of the b vector in each row forming a B matrix.

$$(1) \qquad B_{ij} = b_i / a_{ij} \qquad j = n + 1 \ \dots \ m$$

If B_{hk} is the smallest positive entry in column k then the vector:

$$(2) \qquad R_n = (0 + 0 + .. B_{hk} + 0 + 0 + \dots . \ 0)$$

where there are k - 1 zeroes to the left of the component B_{hk} and n + 1 - k zeroes to the right of it, represents a a vertex of the polyhedron of constraint given by :

$$(3) \qquad A . r \leq b.$$

This vertex vector R_h satisfies the constraint system for it only has to satisfy the members in the k column and it does since it is the smallest term in the k column. It also satisfies any negative term in the k xolumn since a negative term is less than any positive number.

The index for R_h is :

$$(4) \qquad (h, 1, 2, \dots k - 1, \ k + 1, \dots . n)$$

Vector R_h is a solution of the equations whose numbers occur in the index above and by definition is a vertex of the polyhedron of constraint. We state it again:

A vertex of the polyhedron represented by the constraint system

$$(5) \qquad A . r \leq b$$

is given by a vector R which satisfies the constraint system (5) and is a solution of n of the equations in matrix A.

From this one sees that a vertex is given by each column in A which has one or more positive terms in it.

One can then put these vertex vectors into the objective function P and get a series of P values and one can select the vector which gives the largest P value and use this as a new starting point and continue the proces till all neighboring vertices give $c \cdot R$ negative values at which point we have a maximum value of the objective function P.

Knowing how to compute our γ vectors we must now turn our attention to an efficient scheme for solving for the minimum positive t values in the equation:

$$(6) \qquad R = R_0 + t\,\gamma$$

We put (6) into all the equations of A whose numbers do not appear in the index for R and we get a series of positive values of t and we select the smallest from the series which gives a new vertex of the polyhedron when put back into (6). In the same way all the other γ from the index give new vertices on the polyhedron of constraint. n gammas can be computed for each index. In other words each vertex on the polyhedron has n neighbors. Most generally we compute only n-1 γ from each vertex since we already know the γ on which we came to the vertex under consideration. One also uses only those γ for which $c \cdot \gamma$ is positive in maximization.

Example 1. We first do,a simple illustrative numerical example:

Find the maximum value of the objective function:

$$P = 3\,x_1 + 4\,x_2$$

subject to the system of constraints:

$$
\begin{array}{lrcrcr}
(1) & -1\,x_1 & + & 0\,x_2 & \leq & 0 \\
(2) & 0\,x_1 & - & 1\,x_2 & \leq & 0 \\
(3) & 1\,x_1 & - & 2\,x_2 & \leq & 3 \\
(4) & 1\,x_1 & + & 1\,x_2 & \leq & 9 \\
(5) & -3\,x_1 & + & 1\,x_2 & \leq & 1 \\
(6) & 1\,x_1 & + & 2\,x_2 & \leq & 14 \\
((7) & 2\,x_1 & - & 1\,x_2 & \leq & 9 \\
(8) & -2\,x_1 & + & 1\,x_2 & \leq & 2
\end{array}
$$

Every problem in linear programming Mutation-wise will be done by means of a Grand Table into which one inserts the data and cranks out the answer. We shall go to a lot of pains to explain every detail of this Grand Table for it will always be withus in some form. Its organization may be improved but for practical purposes it now seems sufficient for task ahead. We shall never write the first n numbers of the constraint system into the Grand table for they are always the same: a string of - 1,s down the main diagonal and for minimum probs they are + 1,s.

For our simple problemwe write the following Table:

0	1	2	b	1	2	
3	1	- 2	3	3	-	
4	1	1	9	9	9	
5	- 3	1	1	-	1	
6	1	2	14	14	7	
7	2	- 1	9	4.5	-	
8	- 2	1	2	-	2	
x				3	1	
c	3	4		3	4	
P				9	4	
R_3	3	0	9	(3	2)	
γ_+^{-}	2	1				
1	- 2		- 3	3	-	
4	3		3	6	2	
5	- 5		- 9	10	-	
6	4		3	11	2.75	
7	3		6	3	1	
8	- 3		- 6	8	-	
T					1	
9	2	1				
R_7	5	1	19	(7	3)	
γ_+^{-3}	1	2				
1	- 1		- 5	5	-	
2	- 2		- 1	1	-	
4	3		6	3	1	
5	- 1		-1 4	1 5	-	
6	5		7	7	1.4	
8	0		- 9	1 1	+	
T					1	
10	1	2				
R_4	6	3	[illegible]	(4	7)	
γ_+^{-7}	-1	1				
1	1		- 6	6	6	
2	- 1		- 3	3	-	
3	- 3		0	3	-	
5	4		- 15	16	4	
6	1		12	2	2	
8	3		- 9	11	4	
T					2	
11	- 2	2				
R_6	4	5	32	(6	4)	Max.
γ_-^{-4}	-2	1				

This table above is designed for finding t values efficiently. The first row is the numbering of the columns. The 0 column down to 8 contains the numbers of the equations of the constraint system forming the A matrix. The numbers in columns 1 and 2 opposite these numbers are the coefficients of the equations in the A matrix. Column 3 is an empty column for this problem. Column b contains the components of the column vector b. Columns 1 and 2 to the right of column b contain the B matrix whose elements are the components of vector b divided by the elements of A in the same row. The smallest elements in columns 1 and 2 of the B matrix are 3 and 1 respectively. The letter x is written in the same row in column 0. C, the cost vector is written under x and its components 3 and 4 are repeated in the B matrix. P, the objective function, is written under C and the product C . x which is 9 and 4 is written in the same row in the B matrix. 9 is larger than 4 so we take R_3 (3 + 0) as our initial vertex vector. The corresponding value of P = C . R_3 = 9 is written in red in column b. Looking in column 1 in the B matrix we find 3 ang going in the same row to column 0 we find equation . The 3 in R_3 represents equation 3. This same 3 goes into our index (3, 2). In finding R_3 we made x_2 equal to zero and the 2 in the index represents equation (2). We now leave the point (vertex) R_3 and so we compute our leaving direction from our index. The is the column cofactors of the normal of equation (3). The product c. gives a positive value which is indicated by a + subscfipt at . In Column 0 below are the numbers of all the equations of the constraint system not appearing in the index of R_3 . They are: 1, 4, 5, 6, 7, 8. To the right of these numbers in column 1 are the values of put into these equations. In column b to the right of these numbers is the value of R_3 put into these equations. In row 1 to the right of these values is the difference between the last values in the b column and the values in the b column of the same equations in the A matrix. Column 2 to the right of these differences is the ratio of the numbers in column 1 in matrix B to the corresponding numbers in column 1 in matrix A. The smallest number in column 2 of these ratios is 1 and this occurs opposite equation (7). We multiply by this 1 and record it in row 9 under T. We add R_3 and row 9 and get R_7 whose index is (7, 3) and our P = c . R_7 = 19. We repeat the process till we reach R_6 whose index is (6, 4) and P = 32 and = -2 + 1. The - subscript with showing that there are no further increase in P values. In other words we have reached a maximum.

It is instructive to watch the flow of the numbers into and out og the indices.

For a sketch of the constraint polygon of this simple problem see Fig. 15 where we have calculated all its vertices.

Example 2. For a three dimensional problem we write:

Maximize the objective functopn

$$P = 12\,x_1 + x_2 + x_3$$

subject to the constraint system

$$(4) \qquad x_1 + 4\,x_2 - 3\,x_3 \leq 10$$

$$(5) \qquad 5\,x_1 + 6\,x_2 - 8\,x_3 \leq 15$$

$$(6) \qquad x_1 - 3\,x_2 + 4\,x_3 \leq 110$$

$$(7) \qquad x_1 + x_2 - x_3 \leq 4$$

The Grand Table for the solution of this problem is:

0	1	2	3	4	b	0	1	2	3	4
4	1	4	- 3		10		10	2.5	-	
5	5	6	- 8		15		3	2.5	-	
6	1	- 3	4		10		10	-	2.5	
7	1	1	- 1		4		4	4	-	
x							3	2.5	2.5	
C	12	1	1				12	1	1	
P							36	2.5	2.5	
R_5	3	0	0		36		(2,	3,	5)	
γ_4^{-2}	- 6	5	0							
γ_4^{-8}	8	0	5							
1			- 8		- 3	3			-	
4			- 7		3	7			-	
6			28		3	7			1/4	
7			3		3	1			1/3	
T									1/4	
8	2	0	1.25							
R_6	5	0	1.25		61.25		(2,	5,	6)	

γ_{+}^{-2}	0	4	3				
γ_{-}^{-5}	-4	0	5				
1		0		-5	5	+	
3		-3		-1.25	1.25	-	
4		7		1.25	8.75	1.25	
7		1		3.75	0.25	0.25	
T						0.25	
9	0	1	0.75				
R_7	5	1	2	63		(5, 6, 7)	Max.
γ_{-}^{-5}	-1	5	4				
γ_{-}^{-6}	-2	3	1				

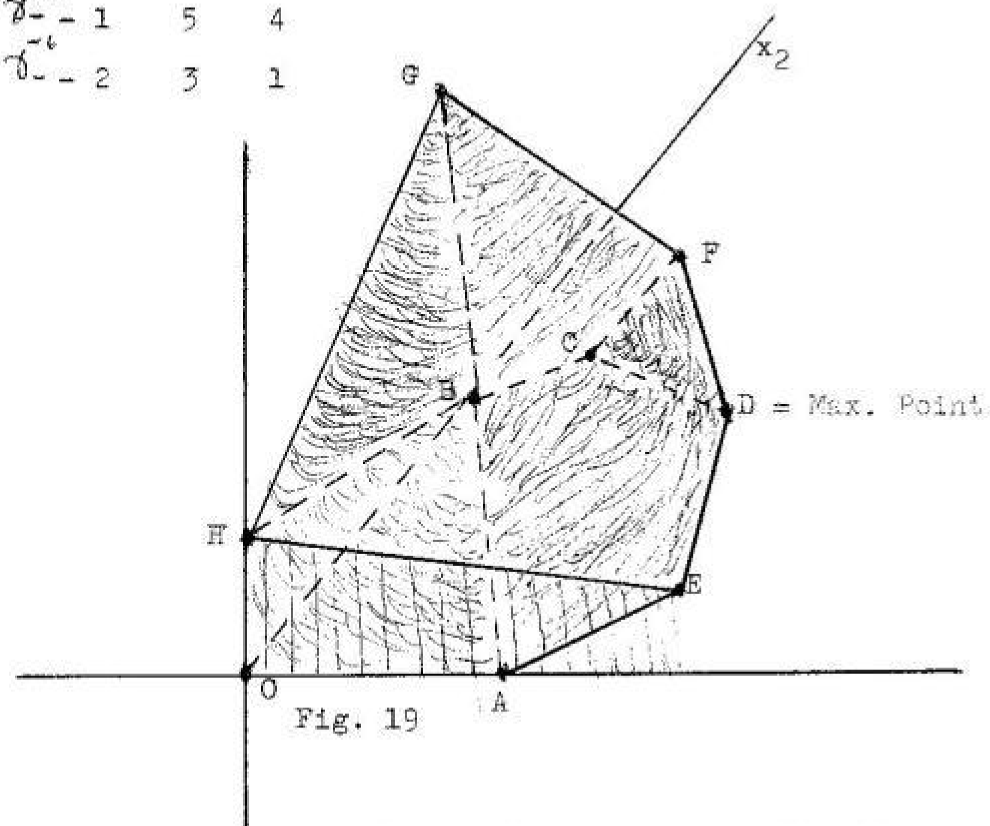

Fig. 19

Above is the view of the polyhedron representing the constraint equations (4), (5), (6), and (7). The path taken by the point in the maximizing process is shown by the red arrows. Just for orientation, we have colculated a number of points so one could see the shape of the polyhedron represented by the constraint equations. They are recorded below.

		P
A	(3 + 0 + 0)	36
B	(0 + 2.5 + 0)	2.5
C	(3 + 4 + 3)	43
D	(5 + 1 + 2)	63 Max.
E	(5 + 0 + 1.25)	61.25
F	(4 + 6 + 6)	60
G	(0 + 10 + 10)	20
H	(0 + 0 + 2.5)	2.5

One knows that point (vertex) D gives a maximum for the two gammas γ^{-5} and γ^{-6} going to the neighbors of D namely C and F give negative values with the cost vector c. The numerical calculation of P for the other vertives of this simple polyhedron show it also. It is good to have a confirmation of a theory.

Example 3. For a four dimensional problem we write:

Maximize the objective function

$$P = 3x_1 + 1x_2 + 2x_3 + 1x_4$$

subject to:

$$
\begin{array}{lrcrcrcrcr}
(5) & x_1 & + & 1x_2 & - & 2x_3 & + & 1x_4 & \leq & 5 \\
(6) & 2x_1 & + & 2x_2 & - & 1x_3 & - & 1x_4 & \leq & 7 \\
(7) & 10x_1 & + & 13x_2 & - & 14x_3 & - & 5x_4 & \leq & 20 \\
(8) & 5x_1 & + & 17x_2 & - & 28x_3 & + & 8x_4 & \leq & 10 \\
(9) & 2x_1 & + & 11x_2 & - & 7x_3 & - & 1x_4 & \leq & 25.
\end{array}
$$

The Grand Table = G. T. is on the following page. We point out again that the first n numbers in any constraint system are always omitted for they are the same for all problems of that dimension. They would only be in the way. One avoids all unnecessary confusion factors that are not needed.

For three or less dimensions one may draw a picture of the constraint polyhedronand a visual picture generally helps but that we cannot draw the polyhedron for higher dimensions will not seriously handicap us, A mental picture is more accurate than one actually drawn.

0	1	2	3	4	b	1	2	3	4	
5	1	1	- 2	1	5	5	5	-	5	
6	2	2	- 1	- 1	7	3.5	3.5	-	-	
7	10	13	-14	- 5	20	2	1.5	-	-	
8	5	17	-28	8	10	2	0.6	-	1.25	
9	2	11	- 7	- 1	25	12.5	2.3	-	-	
x						2	0.6	-	1.25	
C	3	1	2	1		3	1	2	1	
P						6	0.6	-	1.25	
R	2	0	0	0	6	(8	7	2	3	)
σ^{-2}	- 9	5	0	- 5						
σ^{-3}	12	0	5	10						
1	-12				- 2	2	-			
4	-10				0	0	0			
5	12				2	3	$\frac{1}{2}$			
6	9				4	3	1/3			
9	-21				4	21	-			
T							1/4			
10	3	0	1.25	2.5						
R_5	5	0	1.25	2.5	20	(5	8	7	2	)
σ^{-6}	8	0	5	2						
σ^{-7}	- 4	0	1	6						
σ^{-2}	0	4	3	2						
1	- 8		0		- 5	5	-		+	
3	- 5		- 3		- 1.25	1.25	-		-	
4	- 2		- 2		- 2.5	2.5	-		-	
6	9		3		6.25	0.75	0.08		0.25	
9	-21		21		- 1.25	26.25	-		1	
T							0.08		0.25	
11	0.64	0	0.4	0.16						
12	0	1	0.75	0.50						
R_6	5.64	0	1.65	2.66	22.68					
R_6	5	1	2	3	23(6	5	8	7	)	
σ^{-6}	- 3	3	1	-1						
σ^{-5}	2	- 3	- 1	-1						
σ^{-7}	- 1	5	4	4						
1	1				- 5	5	5			
2	- 5				- 1	1	-			
3	- 4				- 2	2	-			
4	- 4				- 3	3	-			
9	21				4	21	1			
T							1			
13	- 1	5	4	4						
R_9	4	6	6	7	37(	9	6	5	8)	
σ^{-9}	- 1	1	1	2						
σ^{-5}	- 2	- 2	- 3	- 5						
σ^{-8}	1	2	3	3						
1	1		- 1		-4	4	4		-	
2	- 1		- 2		- 6	6	-		-	
3	- 1		- 3		- 6	6	-		-	
4	- 2		- 3		- 7	7	-		-	
7	-21		-21		- 1	21	-		-	
T							4			
14	- 4	4	4	8						
R_1	0	10	10	15	45	(1	9	5	8)	Max.

y^{-9}	0	-4	- 3	- 2			
y^{-5}	0	- 4	- 5	- 9			
y^{-8}	0	3	4	5			
2	- 3				- 10	10	-
3	- 4				- 10	10	-
4	- 5				- 15	15	-
6	- 3				- 5	12	-
7	-42				- 85	100	-
T							-

We point out that there are as many initial vertices on the constraint polyhedron as there are columns in the system of constraint. This is also the same as the number of unknowns in the constraint system. For example, one could have started with either of the four columns in the last problem and found an initial vertex. We started with column 1 and got:

$$R_8 = R_7 = 2 + 0 + 0 + 0$$

We could have used column 2 and obtained

$$R_2 = 0 + 0.6 + 0 + 0$$

or column 4 and obtained

$$R_8 = 0 + 0 + 0 * 1.25$$

as initial vertices on the polyhedron of constraint. Here we have omitted their corresponding indices. Column 3 did not give a real vertex since all its entries were negative. For all these vertices the Grand Table will give the same maximum value for the objective function: 45. We choose the first vertex above because its initial value was slightly larger than that of the others and should require less work to reach the maximum.

The same holds true, with some slight changes, for minimum problems. We shall solve a number of illustrative problems incorporating these ideas.

Example. Find the minimum value of the objective function:

$$P = 3\,x_1 + 4\,x_2 + 2\,x_3$$

subject to:

$$\begin{aligned}
x_1 + 3\,x_2 - 2\,x_3 &\geq 6 \\
-3\,x_1 + 5\,x_2 + 5\,x_3 &\geq 15 \\
4\,x_1 + 2\,x_2 - 3\,x_3 &\geq 12 \\
2\,x_1 + 2\,x_2 + 5\,x_3 &\geq 10 \\
x_1 + 3\,x_2 + 6\,x_3 &\geq 6
\end{aligned}$$

0	1	2	3	4	b	1	2	3	4
4	1	3	- 2		6	6	2	-	
5	- 3	5	5		15	-	3	3	
6	4	2	- 3		12	3	6	-	
7	2	2	5		10	5	5	2	
8	1	3	6		6	6	2	1	
x						6	6	3	
C	3	4	2			3	4	2	
P						18	24	6	
R_6	0	6	0		24 (	6	1	3)	
g^{-1}	1	- 2	0	- 5					
g^{-3}	0	3	2	16					
2		- 2			6	- 6	3		
4		- 5			18	-12	2.4		
5		-13			30	-15	1 +		
7		- 2			12	- 2	1		
8		- 5			18	-12	2.4		
T							1		
9	1	- 2	0						
R_7	1	4	0		19 (	7	3	6)	
g^{-3}	8	-13	2	-24					
g^{-6}	1	- 1	0	- 1					
1	8	1			1	- 1	-	-	
2	-13	- 1			4	- 4	4/13	4	
4	-35	- 2			13	- 7	0.2	3.5	
5	-79	- 8			17	- 2	2/79	0.25	
8	-19	- 2			13	- 7	7/19	3.5	
T							2/79	0.25	
10	0.2	- 0.33	0.05						
11	0.25	- 0.25	0						
R_5	1.20	3.67	0.05		16.4 (	5	7	6)	Min.
R_5^2	1.25	3.75	0		16.7 (			)	
g^{-6}	15	25	- 16	113					
g^{-7}	25	-11	26	83					

Note that only those g (gamma) vectors are used which have a negative C.g value which is always recorded in column (n + 1). For a 3rd order problem they are listed in the (3 + 1) or 4th column. The last two gamma have positive C.g values 113 and 83 respectively, showing that there is no more decrease in the value of the objective function which is listed in red in column b to the right of each R. The numbers following each T are the gamma vectors vectors above T multiplied by the t values in row T to the right of T in order. The Resulting vectors are added to the indexed R above to get the various R following these vectors. The best R is selected from these R and a new set of gammas are computed from its index and the process repeated till a minimum is obtained. When the (n + 1) column is needed otherwise, not explainable here, or is not present the C.g values are listed in column b.

This is good and well when some one column in the constraint system has all its entries positive but how about the constraint systems with no such columns? How does one find a vertex on the polyhedron of such a constraint system?

One way is to go with the equations in the constraint system which have positive entries in a column under consideration, testing those equations with negative terms in the column, for satisfaction by the moving point on the reduced polyhedron. One does not compute t values from those equations with negative terms until they come into satisfaction by the moving point on the reduced polyhedron. At the point of satisfaction n gammas must be computed instead of the usual n-1. The process is continued till all the equations with negative terms have been satisfied, the last point being a vertex on the original polyhedron. One can then proceed to the answer certain as previously demonstrated. We shall illustrate with an example or so. We shall re-solve the above problem using column 1 where equation (5) has a - 3 entry. Our Grand Table is the same as the problem just solved:

0	1	2	3	4	B	1	2	3	4	
4	1	3	- 2		6	6	2	-		
5	- 3	5	5		15	-	3	3		
6	4	2	- 3		12	3	6	-		
7	2	2	5		10	5	5	2		
8	1	3	6		6	6	2	1		
X					6	6	3			
C	3	4	2			3	4	2		
P						18	24	6		
R_6	6	0	0		18 (	4	2	3)		
g^{-2}	-3	1	0							
g^{-3}	2	0	1							
1	-3				6	-6	2			
5					-18					
6	-10				24	12	1.2			
7	- 4				12	-2	0.5			
8	0				6	0	7 ?			
T							0.5			
9	- 1.5	0.5	0							
R_7	4.5	0.5	0		15.5(	7	3	4)		
g^{-3}	-19	1	4							
g^{-4}	- 1	1	0							
1	-19				4.5	-4.5	0.24			
2	1				0.5	-0.5	-			
5					-11					
6	-86				19	-7	0.08			
8	8				6	-2	-			
T							0.08			
10	-1.55	0.08	0.33							
R_6	2.95	0.58	0.33		11.83(	6	4	7)		sub-min.
g^{-4}	-8	13	-2							
g^{-7}	1	1	2							

1	8	1		2.95	-2.95	0.37	
2	13	1		0.58	-0.58	-	
3	-2	2		0.33	-0.33	0.16	
5				- 4.30			
8	19	16		6.67	-0.67	-	-
T						0.16	
11	- 1.28	2.08	- 0.33				
R_3	1.67	2.66	0	15.65(3		6	7)
g_2^{-7}	-1	2	0				
g^{-6}	1	- 1	0				
1	-1			1.67	-1.67	1.67	
2	2			2.66	-2.66	-	
4	5			9.65	-3.65	-	
5				8.29			
8	5			9.65	-3.65	-	
T						1.67	
12	-1.67	3.34	0				
R_1	0	6	0	24			
5				30			

Here R_1 satisfies equation 5 and is a point on the original polyhedron. It is the same as the R_6 in the previous calculation. The remaining calculations are the same as those in the first calculation, giving the same minimum answer 18.38.

We now solve the same problem using column 3 where eqs. (4) and (6) have negative entries.

0	1	2	3	4	B	1	2	3	4
4	1	3	- 2		6	6	2	-	
5	- 3	5	5		15	-	3	3	
6	4	2	- 3		12	3	6	-	
7	2	2	5		10	5	5	2	
8	1	3	6		6	6	2	1	
X						6	6	3	
C	3	4	2			3	4	2	
P						18	24	6	
R_5	0	0	3		6 (	5	1	2	) sub-min.
g_2^{-1}	5	0	3						
g^{-2}	0	1	- 1						
3	3	- 1			3	- 3	-	3	
4					- 6				
6					- 9				
7	25	- 3			15	- 5	-	1.67	
8	23	- 3			18	-12	-	4	
T								1.67	
9	0	1.67	- 1.67						
R_7	0	1.67	1.33		9.34(	7	1	5	)
g_1^{-1}	15	25.0	-16						
g^{-5}	0	5	2						

2	25	5		1.67	-1.67	-	-
3	-16	-2		1.33	-1.33	0.08	0.66
4				2.35			
6				- 0.65			
8	- 6	3		12.99	-6.99	1.17	-
T						0.08	0.66
10	1.24	2.07	-1.33				
11	0	3.30	-1.33				
R_3	1.24	3.74	0	18.68	(3	5	7)
R_3	0	4.97	0	19.88			
g^{-5}	- 1	1	0				
g^{-7}	5	3	0				
g^{-3}	-15	- 25	16				
1			- 15	1.24-	1.24		0.083
2			- 25	3.74-	3.74		0.150
4			-132	12.46-	6.46		0.05
6			-158	12.44-	0.44		0.0028
8			6	12.46-	6.46		-
T							0.0028
12	0.04	- 0.07	0.05				
R_6	1.20	3.67	0.05	18.38(	6	5	7) Min.
g^{-5}	-16	26	- 4	+			
g^{-7}	25	- 11	26	+			

Using all three columns of the constraint system we have arrived at the same vertex with the same minimum value namely: 18.38.

In these three solutions one can get some comparison of the amount of work involved for each solution. One can see that for min. problems, when the constraint system contains at least one all positive column , the amount of work is least. In that case min. problems are as easy as the c corresponding max. problems.

3. The Ersatz Function Route to Initial Constraint Polyhedral Vertices.

An Ersatz Function is any equation selected from the c constraint system of a minimization problem. The Ersatz Function is maximized with the reduced constraint system while the minimization of the original objective function is suspended. When the Ersatz Function reaches satisfaction it reenters the original constraint system and the minimization of the original objective function is resumed.

We now solve the same problem above using equation (5) as the Ersatz Function. We use column 1 which has a neg. 3 in equation (5) so that it will not be satisfied by the sub, vector (6 0 0). We shall maximize the left side of equation (5) till it comes into satisfaction either exactly or feasibly. At that point equation (5) re-enters the constraint system and the minimization of the original objective function is resumed. N gamma vectors are computed from the final index in the maximization of an Ersatz Function if it is feasibly satisfied otherwise n - 1 gamma.

If a column has more than one negative entry the maximimization is continued till all equations with negative entries satisfy the reduced constraint system. We shall work this same problem later using column three where we have two negative entries and thus two Ersatz Functions. We get on with the first column illustration now.

0	1	2	3	4	B	1	2	3	4	
4	1	3	-2		6	6	2	-		
5	- 3	5	5		15	-				
6	4	2	- 3		12	3	6	-		
7	2	2	5		10	5	5	2		
8	1	3	6		6	6	2	1		
X						6	6	2		
C	3	4	2			3	4	2		
P						18	24	4		
R_4	6	0	0		- 18	(4	2	3)		
g^{-2}	-3	1	0							
g^{-3}	2	0	1							
1	- 3				6	- 6	2			
6	-10				24	-12	1.2			
7	- 4				12	- 2	0.5			
8	0				6	0	?			
T							0.5			
9	- 1.5	0.5	0							
R_7	4.5	0.5	0		- 11	(7	3	4)		
g^{-3}	-19	1	4							
g^{-4}	- 1	1	0							
1	- 19	- 1			4.5	-4.5	0.24	4.5		
2	1	1			0.5	-0.5	-	-		
6	- 86	- 2			19	-7	0.08	3.5		
8	8	2			6	0	0	0		
T							0.08	3.50		
10	- 1.55	0.08	0.33							
11	- 3.5	3.5	0							
R_6	2.95	0.58	0.33		-4.50					
R_6	1	4	0		17	(6	3	7)		19.

The last R_6 satisfies equation (5) feasibly and thus all the other equations in the constraint system since 17 is greater than 15. One can now resume the the minimization of the original objective function, leaving R_6 in three directions :

$$g^{-6}\ ,\quad g^{-3}\ \text{and}\ \ g^{-7}.$$

The R_6 above is the same as the R_7 in the first solution of this problem. In fact one only has to repeat that problem from R_7 to the end for the solution here. We do it here for completeness.

g^{-3}	8	-131	2	-				
g^{-6}	1	- 1	0	-				
g^{-7}	-1	2	0	+				
1	8	1		1	-1	-	-	
2	-13	-1		4	-4	4/13	4	
4	-35	-2		13	-7	1/5	3.5	
5	-79	-8		17	-2	2/79	1/4	
8	-19	-2		13	-7	7/19	3.5	
T						0.025	0.25	
12	0.20	-0.33	0.05					
13***	0.25	-0.25	0.00					
R_5	1.20	3.67	0.05	18.38(	5	6	7	) Min.
R_5	1.25	3.75	0.00	18.75				
g^{-6}	15	25	-16	+				
g^{-7}	25	- 11	26	+				

Notice that equation (5) entered the constraint system for the first time in the last iteration , having served as the objective function (ersatz function) till satisfaction was reached at 17 which is greater than its constraint value of 15. At this point (5) re-intered the original constraint system. We point out again that three gammas had to be computed at the point of satisfaction instead of the usual two.

In all four solutions the same min. value 18.38 was obtained, lending credence to the correctness of the process.

This process can be repeated when there are more than one negative entry in a given column.

4. Gamma Vectors

In chapter three we discussed the meaning of the gamma vectors. There we used column cofactors to compute them. For large systems this mode of computing them could entail considerable work. We need an alternative to column cofactors which should be efficient and relatively easy if possible. We have devised several schemes for their computation. We shall soon illustrate one of them.

The vector normals to the hyperplanes in a point index will always be designated by:

$$R\ (\ a,\ b,\ c,\ d,\ \ldots .\ )$$

where R is the vector to the point (vertex) determined by the hyperplanes whose normals are a, b, c, d, ...

We assume that a is the last plane arrived at by some gamma. It will remain in the index while the other normals are taken out one at a time and a gamma computed for each of the n - 1 vectors left in the index, each gamma giving the direction to a neighboring vertex on the polyhedron of constraint. If we take b out the resulting gamma will be written:

$$g^{-b}$$

and in like manner for the others:

$$g^{-c},\ g^{-d},\ g^{-e}\ldots\ldots\ldots\ldots$$

There will be n - 1 gamma for each vertex on the polyhedron of constraint since one already knows the vector arriving at a given plane, say, a.

Suppose we have a point R (a, b, c) determined by three planes whose normals are:

$$\begin{array}{llllll} a & = & 2 & + 3 & + 1 \\ b & = & 1 & + 2 & - 1 \\ c & = & -3 & + 1 & + 2 \end{array}$$

$$g^{-b} = \begin{array}{lll} 2 & + 3 & + 1 \\ -3 & + 1 & + 2 \end{array} = 5 - 7 + 11$$

$$g^{-c} = \begin{array}{lll} 2 & + 3 & + 1 \\ 1 & + 2 & - 1 \end{array} = -5 + 3 + 1$$

By actual trial

$$a \cdot g^{-b} = a \cdot g^{-c} = c \cdot g^{-b} = b \cdot g^{-c} = 0$$

showing that each gamma is perpendicular to those vectors l left in the index. This is important. The gammas are easy to compute wgen we have only two vectors with three components each; simply take column cofactors in order from left to right. Note that vector a stayed in the index.

If one had a point index R (a, b, c, d) containing 4 vectors with 4 components each where

$$\begin{array}{llllll} a & = & 2 & + 3 & + 1 & + 1 \\ b & = & 1 & + 2 & * 2 & - 1 \\ c & = & 4 & + 1 & - 3 & + 2 \\ d & = & 3 & - 1 & + 2 & + 1 \end{array}$$

$$g^{-b} = \begin{array}{llll} 2 & + 3 & + 1 & + 1 \\ 4 & + 1 & - 3 & + 2 \\ 3 & - 1 & + 2 & + 1 \end{array}$$

Eliminating the right hand components from two of these vectors in pairs we get

$$\begin{array}{lll} 0 & + 1 & + 1 \\ 1 & - 4 & + 1 \end{array} = 5 + 1 - 1$$

Put this partial g^{-b} into the first three terms of either a, c, or d, say c, and we get:

20 + 1 + 3 = 24. We then divide this 24 by the last 2 in c and we get 24/2 = 12. We then subtract this 12 from our partial gamma and gt the whole gamma:

$g^{-b} = 5 + 1 - 1 - 12.$

By actual trial

$a \cdot g^{-b} = c \cdot g^{-b} = d \cdot g^{-b} = 0$

showing that g^{-b} is perpendicular to a, c, and d, the vectors left in the index. It would have been slightly easier to have put our partial gamma into either a or d since it is easier to divide by 1 than any other number.

The process illustrated above is a universal pattern = U P.

If we write a b for the elimination of the right hand components from a and b we note that

$$g^{-b} = \begin{matrix} a\ c \\ a\ d \end{matrix}, \qquad g^{-c} = \begin{matrix} a\ b \\ a\ d \end{matrix}, \qquad g^{-d} = \begin{matrix} a\ b \\ a\ c \end{matrix}.$$

Note that g^{-b} and g^{-c} have the common element a d, and that the elements of g^{-d} are known from those of g^{-b} and g^{-c} and thus there is no computation required for g^{-d}. One has only to pich up the parts from g^{-b} and g^{-c} and back substitute.

When one is dealing with a large number of variables there are many identical parts which only have to be computed once. In fact the last gamma in any index never has to be computed, only piched up from fore computations. Only one gamma in any index has to be entirely computed,; the computations in the others become less and less till none for the last gamma. We shall illustrate it with a number of numerical examples. Before we illustrate this computation we should like to point out here that large square matrices may be inverted with relatively little computation by means of the gammas. Later we shall illustrate it also.

For gamma computation we start with 4 dimensions.

$$R\,(a, b, c, d)$$

$$\begin{aligned} a &= 2 + 3 + 1 - 1 \\ b &= 1 - 1 + 2 - 2 \\ c &= 3 + 2 - 1 + 1 \\ d &= -1 + 1 + 3 + 1 \end{aligned}$$

$$\begin{aligned} a\ c &= 1 + 1 + 0 \\ a\ d &= 1 + 4 + 4 \end{aligned} \quad = \quad 4 - 4 + 3$$

Put this partial gamma into d and we get $-4 - 4 + 9 = 1.$

$1/1 = 1.$ Subtract this 1 from the partial gamma above andget

$g^{-b} = 4 - 4 + 3 - 1.$

F$\hat{o}$r g^{-c} we write

$$\begin{array}{lcl} a\ b & = & 3 + 7 + 0 \\ a\ d & = & 1 + 4 + 4 \end{array} \quad = 28 - 12 + 5$$

Put this partial gamma into d and get $- 28 - 12 + 15 = - 25$ Divide this $- 25$ by the last 1 in d and get $- 25/1 = - 25$. Subtract this last $- 25$ from the partial gamma above and get the whole gamma:

$g^{-c} = 28 - 12 + 5 + 25.$

For g^{-d} we write:

$$\begin{array}{lcl} a\ b & = & 3 + 7 + 0 \\ a\ c & = & 1 + L + 0 \end{array} \quad = 0 + 0 - 1$$

Put this partial gamma into the first three terms of c and get $0 + 0 + 1 = 1$. Divide this last 1 by the last 1 in c and get $1/1 = 1$. Subtract this last 1 from the partial gamma above and get:

$g^{-d} = 0 + 0 - 1 - 1.$

By actual trial

$a \cdot g^{-b} = c \cdot g^{-b} = d \cdot g^{-b} = 0$

$a \cdot g^{-c} = b \cdot g^{-c} = d \cdot g^{-c} = 0$

$a \cdot g^{-d} = b \cdot g^{-d} = c \cdot g^{-d} = 0$

Note that we computed only one initial row for g^{-c} and none for g^{-d}. We only had to pick up the previously computed elements for g^{-d}. We now do a fifth order gamma computation.

R (a, b, c, d, e)

$$\begin{array}{lcl} a & = & 2 + 3 + 1 - 1 + 2 \\ b & = & 1 - 1 + 2 - 2 - 1 \\ c & = & 3 + 2 - 1 + 1 + 2 \\ d & = & - 1 + 1 + 3 + 1 - 3 \\ e & = & 3 - 1 + 2 + 4 + 1 \end{array}$$

$$\begin{array}{lcl} a\ c & = & - 1 + 1 + 2 - 2 \\ a\ d & = & 4 + 11 + 9 - 1 \\ a\ e & = & - 4 + 5 - 3 - 9 \end{array} \qquad \begin{array}{lcl} b_{12} & = & - 9 - 21 - 14 \\ b_{22} & = & - 1 - 1 + 14 \end{array} =$$

$= - 130 + 58 - 3$. Put this last partial into a d and 91. Then $91/-1 = - 91$. Subtract this from the first partial and get $- 130 + 58 - 3 + 91$. Put this indo e and $- 90$. Then $- 90/1 = - 90$. Subtract this from the second partial and get the total gamma:

$$g^{-b} = -130 + 58 - 3 + 91 + 90.$$

Herw b_{12} and b_{22} are the first and second rows in the second order of differences.

For g^{-c} we write:

$$\begin{aligned} a\,b &= 4 + 1 + 5 - 5 \\ a\,d &= 4 + 11 + 9 - 1 \\ a\,e &= -4 + 5 - 3 - 1 \end{aligned}$$

$$\left.\begin{aligned} c_{12} &= -16 - 54 - 40 \\ c_{22} &= 56 - 16 + 60 \end{aligned}\right\} = 97 + 32 - 82$$

Put the last partial into $a\,d$ and get 2; then $2/-1 = -2$ then our second partial becomes $97 + 32 - 82 + 2$. Put hhis second partial into e and get

$$g^{-c} = 97 + 32 - 82 + 2 - 103.$$

Notice that the last two gammas have the common elements $a\,d$ and ae.

For g^{-d} we write

$$\begin{aligned} a\,b &= 4 + 1 + 5 - 5 \\ a\,c &= -1 + 1 + 2 - 2 \\ a\,e &= -4 + 5 - 3 - 9 \end{aligned}$$

$$\left.\begin{aligned} d_{12} &= 13 - 3 + 0 \\ c_{22} &= 14 - 4 + 15 \end{aligned}\right\} = -9 - 39 - 2$$

from which we get, exactly as in the first two gammas,

$$g^{-d} = -9 - 39 - 2 - 17 + 60.$$

For g^{-e} we write:

a b
a c
a d

$$\left.\begin{aligned} e_{12} &= d_{12} = 13 - 3 + 0 \\ e_{22} &= c_{12} = 8 + 27 + 20 \end{aligned}\right\} = -12 - 52 + 75$$

$$g^{-e} = -12 - 52 + 75 + 55 + 80.$$

By actual trial:

$$
\begin{array}{lllll}
a \cdot g^{-b} = & c \cdot g^{-b} = & d \cdot g^{-b} = & e \cdot g^{-b} = & 0 \\
a \cdot g^{-c} = & b \cdot g^{-c} = & d \cdot g^{-c} = & e \cdot g^{-c} = & 0 \\
a \cdot g^{-d} = & b \cdot g^{-d} = & c \cdot g^{-d} = & e \cdot g^{-d} = & 0 \\
a \cdot g^{-e} = & b \cdot g^{-e} = & c \cdot g^{-e} = & d \cdot g^{-e} = & 0.
\end{array}
$$

Note that we did no calculations for the elements of g^{-e}, only back substitution, and less and less for suceeding gamas.

It seems preferable to compute all the gammas for any given vertex since they cover a lot of the polyhedron of constraint and become progressively easier than to calculate gammas till one reaches an improved point and then compute gammas for it till one reaches an improved point and continue this till no points are left for improvement. Some experimentation here is in order and each can have the fun of doing his own. We now illustrate with the solution of a 6 th order problem.

Minimize $\quad F = C \cdot r$

subject to the constraint system

$$A \cdot r = b$$

$C = 3 + 1 + 2 + 4 + 1 + 3$ row

$r = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ row

$b = 24 + 18 + 12 + 15 + 42 + 16 + 10 + 14$ column

$$
A = \begin{array}{rrrrrr}
1 & 2 & -4 & 1 & 3 & 8 \\
2 & 1 & 3 & -2 & 3 & 6 \\
4 & 3 & -2 & 1 & 2 & 6 \\
-1 & 5 & 1 & 3 & -1 & 15 \\
-3 & 7 & 3 & -1 & 6 & 14 \\
1 & 4 & 1 & 1 & 1 & 8 \\
-2 & 1 & -1 & 2 & 5 & 5 \\
1 & 1 & 2 & 1 & 2 & -1
\end{array}
$$

All $x_i = 0$

On the following pages the Grand Table (G. T.) for the solution of the problem stated above is written. We note that in the Grand Table the numbering of the rows in A start with $n + 1$ since rows down to $n + 1$ are the same fot all nth order matrices and it simplifies the Grand Table.

0	1	2	3	4	5	6	B	1	2	3	4	5	6
7	1	2	- 4	1	3	8	24	24	12	-	24	8	3
8	2	1	3	- 2	3	6	18	9	18	6	-	6	3
9	4	3	- 2	1	2	6	12	3	4	-	12	6	2
10	- 1	5	1	3	- 1	15	15	-	3	15	5	-	1
11	- 3	7	3	- 1	6	14	42	-	6	14	-	7	3
12	1	4	1	1	1	8	16	16	4	16	16	16	2
13	- 2	1	- 1	2	5	5	10	-	10	-	5	2	2
14	1	1	2	1	2	- 1	14	14	14	7	14	7	-
X								24	18	16	24	16	3
C	3	1	2	4	1	3		3	1	2	4	1	3
P								72	18	32	96	16	9
R_8	0	18	0	0	0	0	36 (	8	1	3	4	5	6)
g^{-1}	1	- 2	0	0	0	0	+						
g^{-3}	0	- 3	1	0	0	0	-						
g^{-4}	0	- 2	0	1	0	0	+						
g^{-5}	0	- 3	0	0	1	0	-						
g^{-6}	0	- 6	0	0	0	1	-						
2	- 2	- 3	- 6					18	- 18	9	6	3	
7	-10	- 3	- 4					36	- 12	1.2	4	3	
9	-11	- 7	-12					54	- 42	4	6	3.5	
10	-14	-16	-15					90	- 75	5	4+	5	
11	-20	-15	-28					126	- 84	4	5+	3	
12	-11	-11	-16					72	- 56	5	5+	3.5	
13	- 4	2	- 1					18	- 8	2	-	8	
14	- 1	- 1	- 7					18	- 4	4	4	0.57	
T										1.2	4	0.57	
15	0	- 3.6	1.2	0	0	0							
16	0	-12	0	0	4	0							
17	0	- 3.42	0	0	0	0.57							
R_7	0	14.4	1.2	0	0	0	16.8						
R_7	0	6	0	0	4	0	16 (	7	8	1	3	4	6)
R_{14}	0	14+	0	0	0	0.6	16.3						

****

g^{-8}	0	- 3	0	0	2	0	-						
g^{-1}	1	1	0	0	- 1	0	+						
g^{-3}	0	21	3	0	-10	0	+						
g^{-4}	0	- 9	0	3	5	0	+						
g^{-6}	0	- 6	0	0	- 4	0	-						
2	- 3	- 6					6	- 6		2	1		
5	2	- 4					4	- 4		-	1		
9	- 5	- 8					26	- 14		3	7/4		
10	-17	19					26	- 11		11/17	-		
11	- 9	-24					66	- 24		8/3	1		

12	- 10	- 4					28	- 12		1.2	3		
13	7	-11					26	- 16		-	1+		
14	1	-17					14	0		0	0		
T										0	0		
R_{14}	0	6	0	0	4	0	10 (	14	7	1	3	4	6)
g^{-7}	0	2	0	0	-1	0	+						
g^{-1}	1	1	0	0	-1	0	+						
g^{-3}	0	14	1	0	- 8	0	+						
g^{-4}	0	1	6	1	- 1	0	+						
g^{-6}	0	-19	0	0	10	1							
2	-19						6	- 6		0.33			
5	10						4	- 4		-			
8	17						10	8		0.50			
9	-31						26	-14		0.45			
10	-90						26	-11		0.12			
11	-59						66	-24		0.41			
12	-58						28	-12		0.20			
13	36						26	-16		-			
T										0.12			
18	0	-2.32	0	0	1.22	0.12	Min.						
R_{10}	0	3.68	0	0	5.32	0.12	9.27(10		14	7	1	3	4)
g^{-14}	0	53	0	0	10	-17	+						
g^{-7}	0	-29	0	0	20	11	+						
g^{-1}	18	- 1	0	0	- 8	1	+						
g^{-3}	0	-24	90	0	70	79	+						
g^{-4}	0	- 9	0	10	0	1	+						

The last answer is a min. because the product of all the gammas with C give + values. This problem has several ramifications and one can learn a lot from it by studying them.

In the first index of this problem one could have replaced the 5 by 14 instead of the 7 which we took and one could then proceeded to the same answer by a slightly different route. Multiple T values represent a degenerate polyhedron but it really causes no complication in that one can go any route to the final answer. In a degenerate polyhedron two or more vertices coincide, the number of vertices coinciding at any one vertex being equal to the number of equal minimum T values in the corresponding column of the constraint matrix, not in the index under consideration. It would be good instructive practice for the reader to solve this problem using the substitution 14 sugested above since the answer is known from the present solution using the 7 instead of the 14. Nothing helps like actual practice .

5. Matrix inversion

Given a matrix

$$A = \begin{matrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & ,,, & a_{nn} \end{matrix}$$

We write.: R (a, b, c, d, .. n)

where a, b, c, ... n are the first, second, third, .. n th columns of A. Then

g^{-a}, g^{-b}, ... g^{-n}

are perpendicular to all the columns of which it is not a member. Then

$g^{-a}/ a.g^{-a}$ is a unitary vector and in like mannar for the other vectors. Thus the matrix with rows :

$$\begin{matrix} g^{-a}/ a.g^{-a} \\ g^{-b}/ b.g^{-b} \\ g^{-c}/ c.g^{-c} \\ \cdots \\ g^{-n}/ n.g^{-n} \end{matrix} = A^{-1}$$

is the inverse of matrix A. We do an example:

$$A = \begin{matrix} 2 & 3 & 1 & 4 \\ 1 & -2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ -1 & 1 & -2 & -1 \end{matrix}$$

$$\begin{matrix} a & = & 2 & 1 & 3 & -1 \\ b & = & 3 & -2 & -1 & 2 \\ c & = & 1 & -1 & 2 & -2 \\ d & = & 4 & 3 & 1 & -1 \end{matrix}$$

g^{-a} = 0 0 1 1

g^{-b} = 1 -1 0 1

g^{-c} = 1 -7 -6 -23

g^{-d} = 5 7 -9 -10

$a.g^{-a} = 2$, $b.g^{-b} = 6$, $c.g^{-c} = 42$, $d.g^{-d} = 42$ then

$$A^{-1} = \begin{array}{l} (\ 0 \quad 0 \quad 1 \quad 1\ )/2 \\ (\ 1 \ -1 \quad 0 \quad 1\ )/6 \\ (\ 1 \ -7 \ -6 \ -23\ )/42 \\ (\ 5 \quad 7 \ -9 \ -10\ )/42 \end{array}$$

By actual multiplication one gets:

$$A.\ A^{-1} = \begin{array}{l} (\ 1 \quad 0 \quad 0 \quad 0 \\ (\ 0 \quad 1 \quad 0 \quad 0 \\ (\ 0 \quad 0 \quad 1 \quad 0 \\ (\ 0 \quad 0 \quad 0 \quad 1 \end{array} = A^{-1}.\ A$$

Usually the inverse is written with a common denominator, the determinant of the matrix, but this is not necessary.The determinant of the matrix above is 84. To put ours into the usual shape one could multiply by 84/84.

We point out again that the amount of work in computing

g^{-a}, g^{-b}, g^{-c}, ... g^{-n} diminishes as one goes till

the last one g^{-n} only requires back substitution for its c computation. This is a great saving of labor. Observe it.

In the problems thus far solved the constraint system has had either a greater than or less than according to whether one was minimmizing or maximizing the objective function. We now deal with the sign of equality:

Minimize the objective function:

$P = C . r$

subject to the system of constraint

$A . r = b$

where r is n dimensional and the constraint system contains m equations where m is less than n.

To get an initial solution to the problem one sets n - m of the variables equal to zero and solves the resulting m equations in m variables getting r_0 which contains n - m zeroes in its components. One then omits a zero in tern from the n - m zeroes and calculates the corresponding gammas.

A solution of the system is:

$$r_j = r_0 + t_j \, g^{-j}$$

where g^{-j} is the gamma when the j th variable is ste equal to zero. The equation above is put ito each of the m zero equations outside the index, which always contains the m equations, and the smallest t value selected. A smallest t is selected for each gamma in tern. From these the best P value is determined. The process is repeated till the min. is obtained. Some of the components of r_0 may be negative but this offers no serious dificulty.

When the system has one more columns than rows one can give a mush simpler solution. We do an example of such a system: Maximize the objective function

$$(1) \quad P = C \cdot r$$

subject to

$$(2) \quad \begin{aligned} a \cdot r &= 15 \\ b \cdot r &= 20 \\ d \cdot r &= 10 \end{aligned}$$

$$\begin{aligned} C &= 1 + 2 + 3 - 1 \\ a &= 1 + 2 + 3 + 0 \\ b &= 2 + 1 + 5 + 0 \\ d &= 1 + 2 + 1 + 1 \\ r &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

We take a gamma of a, b, and d and get

$$g = 7 + 1 - 3 - 6$$

We set x_1 in (2) equal to zero and solve the resulting system getting:

$$(3) \quad r_0 = 5/7(0 + 3 + 5 + 3).$$

A general solution for (2) is $r = r_0 + t\,g$

$$(4) \quad r = (5/7)(7t + (t + 3) + (5 - 3t) + (3 - 6t)).$$

$$(5) \quad P = C \cdot r = (5/7)(6)(t + 3) \doteq (5/7)(6)(\tfrac{1}{2} + 3)$$

$$P = 15 = \text{maximum}$$

From (5) P is a maximum when t is a maximum. (4) gives $t = \frac{1}{2}$ as max. t in order that all components of r be positive. With this value of t our vector, from (4), becomes:

$$(6) \quad r = 2.5 (1 + 1 + 1 + 0).$$

All systems with one more columns than rows or one more rows than columns can be solved in this manner. This is unique with Mutation Geometry.