Chapter 5.

1 Duality

In finding the minimum of an objective function with more columns than rows in its constraint system it is convenient to use the notion of Duality.

Find the minimum of an objective function

(1) $P = C \cdot r$

subject to the constraint system

(2) $A \cdot r = b$

where all components of r are required to be positive. The dual may be written: Find the maximum of the objective function

(3) $S = b \cdot U$

subject to the constraint system

(4) $\breve{A} \cdot U = C$

where $\tilde{A}$ is the transpose of A that is A with its rows and columns interchanged. One could write (4) as

(5) $U \cdot A = C.$

Multiply (5) on the right by r dot and get

(6) $U \cdot A \cdot r = C \cdot r = P$

Multiply (2) on the left by U dot and get

(7) $U \cdot A \cdot r = b \cdot U = S.$

The left sizes of (6) and (7) are equal in the limit and thus

(8) $P = S$

or $\text{Min } P = \text{Max } S$

For problems where the min. constraint has far more columns than rows the dual S is much easier to solve than the Primal P as we shall illustrate with numerical examples and otherwise.

After one has found U then S of the Dual one has to find the corresponding r of the Primal P, which is equal to the S of the Dual. We shall find r for a number of illustrative examples. It is always easy to get on a vertex of the polyhedron of constraint in problems of maximization.

In ordinary minimization problems where the number of rows and columns in the constraint system are not too different and where at least one column has all positive entrants it is easier to go with the Primal and then one gets the r of the Primal in the process. Even when there is no column with all positive entrants one may want to go with the Primal, using an (ersatz) function which will be defined later. A lot of practice will help one to make the choice between the Primal and Dual. Experience is a great teacher.

Henceforth P will be used as the primal objective function and S that of the dual; r the primal vector and U that of the dual.

C is the cost vector in the orimal system and b, from the system, is the cost vector in the dual. This is obvious from equation (3).

Find the minimum value of the objective function:

$$P = 3x_1 + 9x_2 + 9x_3 + 14x_4 + 2x_5 + x_6$$

subject to the constraint system

(7) $x_1 + 2x_2 + x_3 + x_4 - 2x_5 - 3x_6 = 3$

(8) $-2x_1 - x_2 + x_3 + 2x_4 + x_5 + x_6 = 4$

The mutation table following gives :

$$r = 0 + 0 + 2 + 1 + 0 + 0$$

$$P = C \cdot r = 32 = \text{min.}$$

$$A = \begin{matrix} 1 & + 2 & + 1 & + 1 & - 2 & - 3 \\ -2 & - 1 & + 1 & + 2 & + 1 & + 1 \end{matrix}$$

$$A = \begin{matrix} 1 & - & 2 \\ 2 & - & 1 \\ 1 & + & 1 \\ 1 & + & 2 \\ -2 & + & 1 \\ -3 & + & 1 \end{matrix} \qquad b = \begin{matrix} 3 \\ 4 \end{matrix}$$

0	1	2	3	4	5	6	B	1	2	3	4	5	6
7	1	2	1	1	- 2	- 3	3	3	1.5	3	3	-	-
8	- 2	- 1	1	2	1	1	4	-	-	4	2	4	4
X								0	0	4	3	0	0
C	3	9	9	14	2	1		3	9	9	14	2	1
P								0	0	36	42	0	0
r_8	0	0	4	0	0	0	36 (	8	1	2	4	5	6)
g^{-1}	1	0	2	0	0	0	+						
g^{-2}	0	1	1	0	0	0	+						
g^{-4}	0	0	- 2	1	0	0	-						
g^{-5}	0	0	- 1	0	1	0	-						
g^{-6}	0	0	- 1	0	0	1	-						
3	- 2	- 1	- 1				4	- 4	2	4	4		
7	- 1	- 3	- 4				4	- 1	1	1/3	¼		
[illegible]									1	1/3	¼		
9	0	0	- 2	1	0	0							
10	0	0	- 1/3	0	1/3	0		Min.					
11	0	0	- ¼	0	0	¼							
r_7	0	0	2	1	0	0	32(	7	8	6	5	1	2)
r_7	0	0	11/3	0	1/3	0	[illegible]						
r_7	0	0	15/4	0	0	½	34						
g^{-8}	0	0	- 1	1	0	0	+						
g^{-6}	0	0	7	- 4	0	1	+						
g^{-5}	0	0	5	- 3	1	0	+						
g^{-2}	0	1	- 5	3	0	0	+						
g^{-1}	1	0	- 4	3	0	0	+						

The dual is: maximize the objective function

$$S = b \cdot U$$

subject to:

$$\begin{aligned}
&(3) & U_1 - 2\,U_2 &\leq 3 \\
&(4) & 2\,U_1 - 1\,U_2 &\leq 9 \\
&(5) & U_1 + 1\,U_2 &\leq 9 \\
&(6) & U_1 + 1\,U_2 &\leq 14 \\
&(7) & -2\,U_1 + 1\,U_2 &\leq 2 \\
&(8) & -3\,U_1 + 1\,U_2 &\leq 1
\end{aligned}$$

The dual solution is easy:

$U = 4 + 5$, index (5, 6)

$S = b \cdot U = (3 + 4) \cdot (4 + 5) = 12 + 20 = 32$

$P = C \cdot r = 18 + 14 = 32$

$P(\min) = S(\max) = 32$

For a sketch of the polyhedron of constraint of the max. of this problem see Fig. 15 where all its vertices are calculated.

We notice that r_7 in the G. T. has just two components that are non-zero which is the same as the number of rows in the A matrix, namely 2. Is the number of non-zero components in the min vector always equal to the number of rows in the A matrix? We also notice that the numbers in the final index for U are associated with the C components 9 and 14. From the G. T. we read the min primal vector:

$r_7 = 0 + 0 + 2 + 1 + 0 + 0$

We wonder whether we can read the r vector from the solution of the dual and vice versa without an actual solution. We shall soon look into this possibility.

One would not want to use the dual on a min problem whose constraint system had more rows than columns.

From observed patterns it seems that r and U final have the same number of non-zero components.

The numbers in the final index of U tell us which components of C they are associated with and thus which components of r are non-zero. With the equation

$$C \cdot R = b \cdot U$$

known we use one or more of the equations below n in the primal as exact and solve for the components of r. We illustrate with the example just worked.

From the index of the dual we have the numbers (5) and (6). Looking in the constraint system of U we see that (5) and (6) are associated with components 9 and 14 of C which are associated with components x_3 and x_4 of r. We may now write:

$$r = 0 + 0 + x_3 + x_4 + 0 + 0$$

$$C.r = 9x_3 + 14x_4 = 32$$

We put the r above into (7) of matrix A and get

$$x_3 + x_4 = 3$$

Solving the last two equations we get $x_3 = 2$, $x_4 = 1$.

$$r = 0 + 0 + 2 + 1 + 0 + 0$$

which agrees with that originally calculated.

If we had used (8) instead of (7) we would have had

$$9x_3 + 14x_4 = 32$$

$$x_3 + 2x_4 = 4$$

from which we again get $x_3 = 2$, $x_4 = 1$

These answers also satisfy (7) and (8).

We now do another illustrative example. Minimize:

$$P = 4x_1 + 2x_2 + 3x_3 + 6x_4$$

subject to

(5) $$x_1 + 3x_2 + 2x_3 - 4x_4 = 12$$

(6) $$-2x_1 + x_2 + 5x_3 + 2x_4 = 10$$

(7) $$3x_1 - 4x_2 + 6x_3 + 8x_4 = 24$$

The Grand Table solution is:

$$r = (0 + 12 + 60 + 0)/13$$

$C . r = 15.7 = P$, index (7, 5, 1, 4)

The dual is: maximize

$$S = b . U = 12u_1 + 10u_2 + 24u_3$$

subject to

$$(4) \quad u_1 - 2u_2 + 3u_3 = 4$$

$$(5) \quad 3u_1 + u_2 - 4u_3 = 2$$

$$(6) \quad 2u_1 + 5u_2 + 6u_3 = 3$$

$$(7) \quad -4u_1 + 2u_2 + 8u_3 = 6$$

A Grand Table solution for this is

$$U = (24 + 0 \quad 5)/26$$

$$S = b \cdot U = 15.7, \quad \text{index } (5, 6, 2)$$

One can now write r as

$$r = 0 + x_2 + x_3 + 0.$$

$$C \cdot r = 2x_2 + 3x_3 = 15.7$$

$$3x_2 + 2x_3 = 12, \quad \text{see } (5) \text{ of A.}$$

The solution of the last two equations is:

$$x_2 = 12/13, \quad x_3 = 60/13 \quad \text{then}$$

$$r = (0 + 12 + 60 + 0)/13$$

agreeing with that previously calculated by the G. T.

We shall now look at the calculation of the r vector from the dual data in a slightly different manner:

Minimize the objective function

$$P = C \cdot r$$

subject to to the constraints

$$A \cdot r \geq b$$

The dual is maximize the objective function

$$S = b \cdot U$$

subject to the constraints

$$U \cdot A \leq C$$

We solve the dualgetting an index which tells us which components of r are non-zero. From this index we can construct a square matrix A_0 formed from the equations whose numbers are in the index. We can then write:

$$r = A_0^{-1} . b$$

From the U index we get A and from this we get A_0. From the U index of the first illustrative example we got

$$(5) \qquad 1\,U_1 + 1\,U_2 = 9$$

$$(6) \qquad 1\,U_1 + 2\,U_2 = 14$$

from which we obtain

$$A_0 = \begin{matrix} 1 & +1 \\ 1 & +2 \end{matrix}, \qquad A = \begin{matrix} 1 & +1 \\ 1 & +2 \end{matrix}$$

$$b = 3 + 4 \qquad A_0^{-1} = \begin{matrix} 2 & -1 \\ -1 & +1 \end{matrix}$$

$$r = A_0^{-1} . b = 0 + 0 + 2 + 1 + 0 + 0$$

which is the same as that calculated in the solution of the primal.

From the U index of the second illustrative example we have U (5, 6, 2)

$$(5) \qquad 3\,U_1 + 1\,U_2 - 4\,U_3 = 2$$

$$(6) \qquad 2\,U_1 + 5\,U_2 + 6\,U_3 = 3$$

$$(2) \qquad 0\,U_1 - 1\,U_2 + 0\,U_3 = 0$$

$$A = \begin{matrix} 3 & +1 & -4 \\ 2 & +5 & +6 \\ 0 & -1 & +0 \end{matrix}, \qquad A = \begin{matrix} 3 & +2 & +0 \\ 1 & +5 & -1 \\ -4 & +6 & +0 \end{matrix}$$

$$A_0^{-1} = \begin{matrix} (\;3 & +0 & -1\;)/13 \\ (\;4 & +0 & +3\;)/26 \\ (\;2 & -2 & +1\;)/1 \end{matrix}, \qquad b = 12 + 10 + 24$$

$$r = A_0^{-1} . b = (\;12 + 60\;)/13 = (\;0 + 12 + 60 + 0\;)/13$$

which is the same as that calculated from the primal.

We only needed the x_2 and x_3 components since the others were 0. In the last problems we have shown how to find the primal min, vector **r** when we had given : A, b, c, U, and the the index of U. Can we find U when we are given A, b, c, r, and the index of **r** ? In the last problem:

$$C = 3 + 9 + 9 + 14 + 2 + 1$$

$$r = 0 + 0 + 2 + 1 + 0 + 0$$

$$A = \begin{matrix} 1 & +2 & +1 & +1 & -2 & -3 \\ -2 & -1 & +1 & +2 & +1 & +1 \end{matrix}$$

$$b = 3 + 4$$

$$C_0 = 9 + 14$$

$$A_0 = \begin{matrix} 1 & +1 \\ 1 & +2 \end{matrix}$$

$$A_0^{-1} = \begin{matrix} 2 & -1 \\ -1 & +1 \end{matrix}$$

$$U = C_0 \cdot A_0^{-1} = 4 + 5$$

which checks with the previous calculation.

The more practical side is to be able to go from U to r since one can always easily get on the polyhedron of constraint in maximization problems.

In the second illustrative problem:

$$C = 4 + 2 + 3 + 6$$

$$r = (0 + 12 - 60 - 0)$$

$$A = \begin{matrix} 1 & +3 & +2 & -4 \\ -2 & +1 & +5 & +2 \\ 3 & -4 & +6 & +8 \end{matrix}$$

$$b = 12 + 10 + 24$$

$$C_0 = 2 + 3$$

$$A_0 = \begin{matrix} 3 & +2 \\ -4 & +6 \end{matrix} \qquad A_0^{-1} = \begin{matrix} 6 & -2 \\ 4 & +3 \end{matrix}$$

$$U = C_0 \cdot A_0^{-1} = (\ 2 + 3\ ) \begin{matrix} 6 & -2 \\ 4 & +3 \end{matrix} = (\ 24 + 5\ )/26$$

$$= (\ 24 + 0 + 5\ )/26.$$

The (5) and (7) correspond to the x_1 and x_3 in the A matrix.

The number of components in U is the same as the number of rows in matrix A. In the last case there are 3 rows in A and thus U must have 3 components. On this topic we do a final illustrative example: Minimize

$$P = 2\,x_1 + 3\,x_2$$

subject to:

(3) $\quad -1\,x_1 + 6\,x_2 = 12$

(4) $\quad 5\,x_1 - 1\,x_2 = 5$

(5) $\quad 1\,x_1 + 1\,x_2 = 7$

(6) $\quad 1\,x_1 + 2\,x_2 = 11.$

The dual is maximize

$$S = 12\,u_1 + 5\,u_2 + 7\,u_3 + 11\,u_4$$

subject to

(5) $\quad -1\,u_1 + 5\,u_2 + 1\,u_3 + 1\,u_4 = 2$

(6) $\quad 6\,u_1 - 1\,u_2 + 1\,u_3 + 2\,u_4 = 3$

The solution for U is

$$U = (\ 0 + 0 + 1 + 1\ ), \text{ Index } (\ 5, 6, 1, 2\ )$$

$$S = b \cdot U = 18$$

$$A_0 = \begin{matrix} 1 & + & 1 \\ 1 & + & 2 \end{matrix} , \qquad A_0^{-1} = \begin{matrix} 2 & - & 1 \\ -1 & + & 1 \end{matrix} \qquad b = \begin{matrix} 7 \\ 11 \end{matrix}$$

$$r = A_0^{-1} . \mathbf{b} = 3 + 4.$$

$$P = \mathbf{c} . r = (2 + 3) . (3 + 4) = 18$$

which is the same as that obtained from the dual. This agreement gives credence to the correctness of the theory.