Chapter 6.

1. Parametric Linear Programming.

The key to parametric linear programming, in accordance with the New Science of Mutation Geometry, is the aggregate of numbers in the point index for an optimal solution. It tells us what hyper planes combine to form this optimal point and it thus shows us the bounds of varation of the objective function if it is to conform to the system of constraints.

The starting point of parametric linear programing is, of course, the initial optimum solution point.

The varation (teetering) of the objective function is to take place about this optimum point.

That varation has a permissable region of validity whether we are dealing with a single or many parameters.

The objective function will be written as a function of the parameters and then its region of validity will be determined. One may then vary the cost coefficients in a way most suitable to fit any production scheduling. We shall get on with the job.

We shall start with the simplest first. We shall use one of the problems already solved in order to keep out any confusion factors that might detract the attention of the reader from the essentials.

We solved the problem: find a vector r which maximizes the objective function

$$P = 3x_1 + 4x_2$$

subject to the constraints:

(3)	x_1	$-$	$2\,x_2$	=	3
(4)	x_1	$+$	$1\,x_2$	=	9
(5)	$-3\,x_1$	$+$	$1\,x_2$	=	1
(6)	x_1	$+$	$2\,x_2$	=	14
(7)	$2\,x_1$	$-$	$1\,x_2$	=	9
(8)	$-2\,x_1$	$+$	$1\,x_2$	=	2

The original solution for this problem may be seen on page 39 and its constraint polygon on page 23, Fig. 15. The solution obtained is:

$$r_0 \; (4 + 5) \quad 32 \quad (6, 4).$$

The numbers 6 and 4 in the index refer now to equations (6) and (4) and not inequalities. The solution of these two equations give the vector r_0 whose components are 4 + 5.

$$P_0 = C \cdot r_0 = (3 + 4) \cdot (4 + 5) = 32.$$

The respective normals of (4) and (6) are:

$$1 + 1 \text{ and } 1 + 2.$$

The normal of our cost vector C can vary (teeter) about the point r_0 (4 + 5) until it coincides with each of the normals of equations (4) and (6). It is required to stay within these limits. We may now write:

$$C = 3 + K$$

Then

$$P = C \cdot r = (3 + K) \cdot (4 + 5)$$
$$= 12 + 5\,K.$$

We now write C as a linear combination of the normals of (4) and (6):

$$h_1 (1 + 1) + h_2 (1 + 2) = 3 + K, \text{ whence}$$

$$h_1 + h_2 = 3$$

$$h_1 + 2h_2 = K$$

$$h_1 = 6 - K = 0, \quad K = 6$$

$$h_2 = K - 3 = 0, \quad K = 3$$

Thus K can vary from 3 to 6 in its teetering about r_0 and still satisfy the system of constraint. The constraint system is satisfied for any values between the limits 3 and 6.

$$P = 12 + 5K = 12 + 5(6) = 42 = \text{max.}$$

$$P = \quad = 12 + 5(4) = 32 = \text{original}$$

$$P = \quad = 12 + 5(3) = 27 = \text{min.}$$

If one had desired to retain both components of C one could write

$$t + 4 = 6, \quad t = 2$$

$$t + 4 = 3, \quad t = -1$$

We shall soon generalize the process to n dimensions with n - 1 parameters but before we do that we shall do a three variable illustrative problem.

A three dimensional problem, see Fig. 19 for its constraint polygon, was solved where

$$P = 12x_1 + x_2 + x_3$$

$$r_0 = 5 + 1 + 2, \quad 63. \quad (5, 6, 7)$$

$$C_0 = 12 + 1 + 1.$$

The normals of the index equations (5), (6), and (7) are:

$$e_1 = 5 + 6 - 8$$

$$e_2 = 1 - 3 + 4$$

$$e_3 = 1 + 1 - 1$$

The gammas of these are

$$e_1^{-1} = 1 - 5 - 4$$

$$e_2^{-1} = 2 - 3 - 1$$

$e_3^{-1} =$ [illegible] $+ 4 + 3$

We next set our original cost vector C_0 to

$$C_0 = C_1 + K_1 + K_2 = 12 + K_1 + K_2$$

$$P_0 = C_0 \cdot r_0 = 60 + K_1 + 2 K_2.$$

We now write

$$p = K_1 + 2 K_2.$$

We now write our new cost vector as a linear combination of the normals of the equations appearing in the optimal index:

$$h_1 e_1 + h_2 e_2 + h_3 e_3 \quad 12 + K_1 + K_2.$$

$$h_1 = C \cdot e_1^{-1} = 12 - 5 K_1 - 4 K_2 = 0$$

$$h_2 = C \cdot e_2^{-1} = 24 - 3 K_1 - 1 K_2 = 0$$

$$h_3 = C \cdot e_3^{-1} = 0 + 4 K_1 + 3 K_2 = 0.$$

One may now state the parametric side of the original equation: Maximize the objective function

$$p = K_1 + 2 K_2$$

subject to the constraints

(1) $5 K_1 + 4 K_2 = 12$

(2) $3 K_1 + 1 K_2 = 24$

(3) $4 K_1 + 3 K_2 = 0$

We only need to deal with equation (1) for (3) is satisfied for all positive values of K_1 and K_2 and all the axial intercepts of (2) are greater than the corresponding ones of (1). The axial vertices of (1) are

$$(K_1, K_2) = (0, 3)$$

$$(K_1, K_2) = (2.4, 0)$$

$$p (0, 3) = 6$$

$$p (1, 1) = 3$$

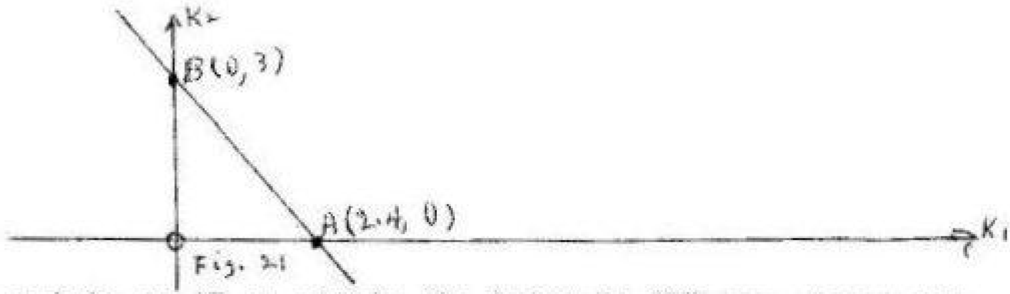
$$p (2.4, 0) = 2.4$$

$$P = 60 + p = 60 + 6 = 66 = \text{new max.}$$

$$P = 60 + p = 60 + 3 = 63 = \text{original max.}$$

$$P = 60 + p = 60 + 2.4 = 62.4 = \text{min.}$$

The shetch for the new configuration is shown below.



All points on AB or within the triangle OAB are acceptable points . They make all h_i positive or make the cost plane teeter on the point R_0 without cutting the constraint polyhedron.

We point out that we have gained a new improved maximum by this teetering of the cost plane on the optimum point R_0. This is significant in thst one can adjust production schedules to fit any cost numbers inside triangle OAB inclusive. This is quite an advantage and convenience. It points out that parametric linear programming is far more general than the usual linear programming.

2 The Gamma Theorem of Polarization

One may find the equations of constraint for the new objective function p from the theory of polarization. The three neighboring points to R_0 are R_1, R_2, and R_3:

$R_1 = 3 + 4 + 3$

$R_2 = 4 + 6 + 6$

$R_3 = 5 + 0 + 1.25$

(1) $P = R_0 . C = 60 + K_1 + 2 K_2$

We now replace R_0 by R_1, R_2, R_3 and get:

(2) $36 + 4 k_1 + 3 K_2 \leqq P$

(3) $48 + 6 K_1 + 6 K_2 \leqq P$

(4) $60 + 0 K_1 + 1.25 K_2 \leqq P$

since each of the points R_1 R_3 is on the near side of the cost plane thru R_0. Subtracting (2) (3) and (4) from (1) we get:

(5) $3 K_1 + 1 K_1 \leqq 24$

(6) $5 K_1 + 4 K_2 \leqq {}^{-}12$

(7) $4 K_1 + 3 K_2 \geqq 0$

These last three equations are the same as those obtained from another viewpoint.

The theory of Polarization is a powerful tool for finding the new equations of constraint in parametric linear programming.

One can also use my unique F theorem to accomplish the same end. The Polarization and and F theorems confirm eachother. We list below some options for C:

$C = 12 + 0 + 3$

$C = 12 + 2.4 + 0$

$C = 12 + 0.2 + 2.75$

$C = 12 + 0.3 + 2.625$

$C = 12 + 0.4 + 2.50$

$C = 12 + 1 + 1.75$

$C = 12 + 2 + 0.50$

$C = 12 + 2.4 + 0$

The range of K_1 is from 0 to 2.4 and that of K_2 0 to 3.

Suppose that the objective function to be maxed is

$$P = C_0 . R_0$$

$$C_0 = C_1 + C_2 + \dots C_n$$

$$C_0 = C_1 + K$$

$$K = 0 + K_1 + K_2 + \dots . K_{n-1}$$

$$P = C_0 . R_0 = (C_1 + K) . R_0 =$$

$$= C_1 . R_0 + K . R_0 = C_1 . R_0 + p.$$

We then maximize :

(1) $$p = R_0 . K.$$

Suppose now that the normals of the n equations in the index of R_0 are

$$e_1 = e_{11} + e_{12} + \dots e_{1n}$$

$$e_2 = e_{21} + e_{22} + \dots e_{2n}$$

$$e_n = e_{n1} + e_{n2} + \dots e_{nn}$$

We now write :

(2) $$h_1 e_1 + h_2 e_2 + \dots h_n e_n = C_0$$

from which we obtain:

$$h_1 = C_0 . e_1^{-1} = e_1^{-1} . (C_1 + K) =$$

$$= C_1 . e_1^{-1} \quad K . e_1^{-1} = 0$$

$$h_2 = e_2^{-1} . (C_1 + K) = C_1 . e_2^{-1} + K.e_2^{-1} = 0$$

$$h_n = e_n^{-1} . (C_1 + K) = C_1 . e_n^{-1} + K.e_n^{-1} = 0$$

Here $$e_j^{-1} = g^{-j} / j . g^{-j}$$

is the reciprocal of e_j, and g^{-j} is the gamma vector of the normal j.in the index of R_0. Since we do not need the denominators in the last equation we can just use the gammas.

We can now pose the problem: Maximize the objective function

$$p = R_0 \cdot K$$

subject to the system of h constraints above. The h constraint parameters keep the new cost vector C teetering about the maximum point R_0. They restrain the normal of C within the polyhedral angle formed by the normals e_j.

This teetering of the cost plane about the max. point R_0 with its normal restrained within a given polyhedral angle is the Cardinal Principle of Parametric linear programming in the styling of Mutation Geometry.

We redo the two dimensional problem last solved where

$$R_0 = 4 + 5$$

$$C_0 = 3 + 4$$

$$e_1 = 1 + 1$$

$$e_2 = 1 + 2$$

$$e_1^{-1} = 2 - 1$$

$$e_2^{-1} = -1 + 1$$

$$C_0 = C_1 + K_1 = 3 + K_1.$$

$$h_1 = e_1^{-1} \cdot C_0 = 6 - K_1 = 0$$

$$K_1 = 6$$

$$h_2 = e_2^{-1} \cdot C_0 = -3 + K_1 = 0$$

$$K_1 = 3$$

$$C = C_1 + K_1 = 3 + 6 = 3(1 + 2)$$

$$C = C_1 + K_1 = 3 + 3 = 3(1 + 1).$$

These last two expressions show us that the teetering plane in its limits coincides with e_1 and e_2 which it should if it is not to cut the constraint polyhedron. We now get

$$P_1 = C \cdot R_0 = (3 + 3).(4 + 5) = 27 = \text{min}$$

$$P_0 = C \cdot R_0 = (3 + 4).(4 + 5) = 32 = \text{orig. max.}$$

$$P_2 = C. R_0 = (3 + 6).(4 + 5) = 42 = \text{max.}$$

The first variant normal of the teetering plane

$$C = 3 + 6 = 3(1 + 2)$$

coincides with the fixed normal $e_2 = (1 + 2)$. The second variant normal of the teetering plane, in the limit, coincides with the normal $e_1 = (1 + 1)$.

$$C = 3 + 3 = 3(1 + 1).$$

The fixed normals are those of equations in the index. These are significant results. The K value varies from 3 to 6. We now redo the third order equation:

$$R_0 = 5 + 1 + 2$$

$$C_0 = 12 + 1 + 1$$

$$C_0 = C_1 + K_1 + K_2$$

$$e_1 = 5 + 6 - 8$$

$$e_e = 1 - 3 + 4$$

$$e_3 = 1 + 1 - 1$$

$$e_1^{-1} = 1 - 5 - 4$$

$$e_2^{-1} = 2 - 3 - 1$$

$$e_3^{-1} = 0 + 4 + 3$$

$$P = R_0 . C_0 = R_0 . C_1 + R_0 . K = R_0 . C_1 + p$$

$$p = R_0 . K$$

$$h_1 = e_1^{-1} . C_0 \geqq 0$$

$$h_2 = e_2^{-1} . C_0 \geqq 0$$

$$h_3 = e_3^{-1} . C_0 \geqq 0$$

$$12 - 5 K_1 - 4 K_2 \geqq 0$$

$$24 - 3 K_1 - 1 K_2 \geqq 0$$

$$0 + 4 K_1 * 3 K_2 \geqq 0$$

From the last three equations we obtain:

$$(1) \qquad 5\,K_1 + 4\,K_2 = 12$$

$$(2) \qquad 3\,K_1 + 1\,K_2 = 24$$

$$(3) \qquad 4\,K_1 + 3\,K_2 = 0$$

Equation (3) is satisfied for all positive values of K_1 and K_2; and equation (2) has all its axial intercepts larger than the corresponding of equation (1). Thus we only have to deal with equation (1). W. now maximize

$$p = R_0 . K = K_1 + 2\,K_2$$

subject to the constraint (1). There are only two points, vertices, on (1) They are:

$$(K_1 , K_2) = (2.4, \ 0)$$

$$(K_1 , K_2) = (0, \ 3)$$

$$p\ (2.4, 0) = 2.4$$

$$p\ (0, \ 3) = 6$$

$P = R_0.C = (5 + 1 + 2).(12 + 2.4 + 0) = 62.4$ min

$P = R_0.C = (5 + 1 + 2).(12 + 1 + 1) = 63$ orig.max.

$P = R_0.C = (5 + 1 + 2).(12 + 0 + 3) = 66$ new max.

See the configuration below.

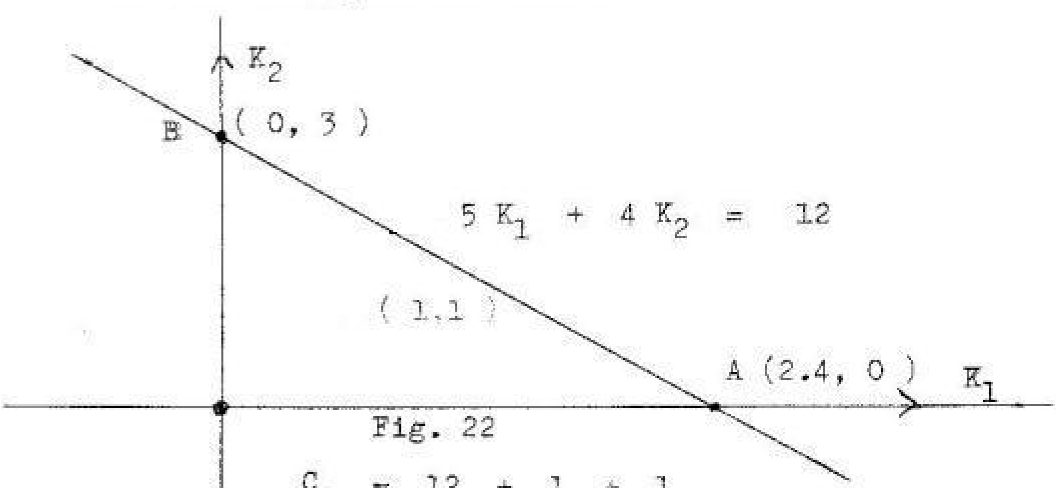


Fig. 22

Below is a list of options for the teetering plane:

C	=	12	+	K_1	+	K_2
				(1	+	1)
				(2.4	+	0)
				(0.2	+	2.75)
				(0.3	+	2.625)
				(0.4	+	2.50)
				(1	+	1.75)
				(2	+	0.50)

All points on the perimeter or within the area OABO are acceptable options.

The Gamma Theorem Of Polarization can be used to check the H Theorem, and vice versa.

3. The H Theorem

The H Theorem states that the teetering cost plane thru the optimum point R_0 cannot cut the constraint polyhedron as long as its normal C is governed by the expression:

$$(1) \qquad h_1 e_1 + h_2 e_2 + \dots h_n e_n = C$$

where the e_j are the normals to the faces of the planes whose numbers occur in the index for R_0 while all h_j are positive.

This H theorem is one of the foundations for n-1 parameter linear programming for an n dimensional space.

Such does not seem hitherto to be known in the literature. We shall prove it. We first make the truth of it plausible with a simple two dimensional diagram. The present diagram refers to a max. problem in two dimensions. See the sketch for Fig. 23 below. The theorem forces the new cost plane C to teeter about the maximum point R_0 without cutting the original polyhedron of constraint. One will see its beauty and power as we proceed.

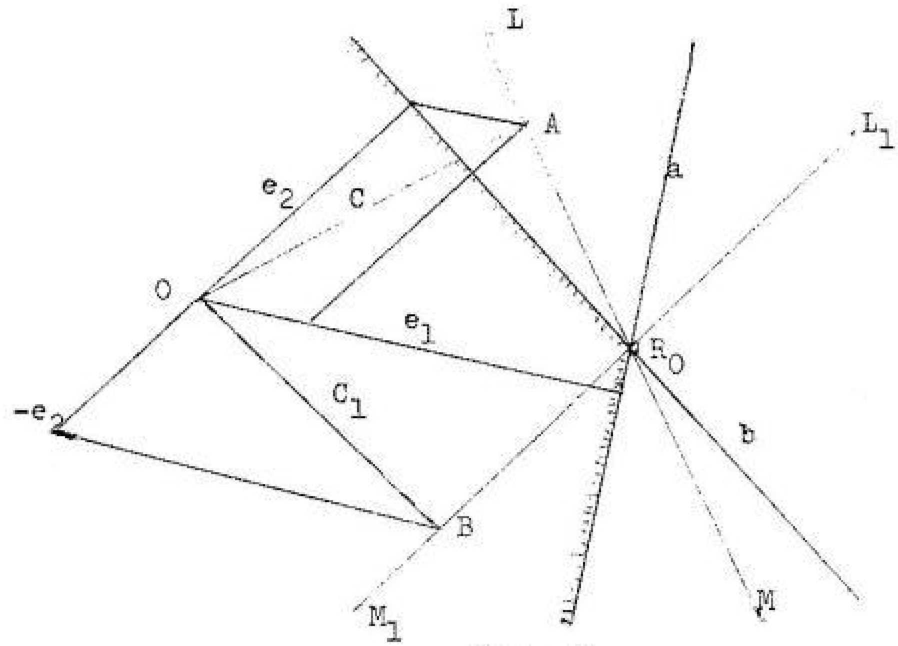


Fig. 23

Let lines a and b be the two lines of a polygon of constraint meeting in the optimal point R_0 and let LM be the cost plane thru R_0 and C its normal from point O. Let e_1 and e_2 be the normals to a and b respectively. As drawn, one sees that C lies between e_1 and e_2 and we may write

$$(2) \qquad C = h_1 e_1 + h_2 e_2$$

where h_1 and h_2 are both positive. Now rotate LM about R_0 till it passes inside the polygon as at $L_1 M_1$ then C, its normal, may be written

$$(3) \qquad C_1 = h_1 e_1 - h_2 e_2$$

showing that as long as it stays inside the hatched angle a b no change in the sign of h occurs.

This is a simple visual illustration of the M theorem. Its truth is very plausible and even self evident for this simple case. For three dimensions we write:

$$(4) \qquad C = h_1 e_1 + h_2 e_2 + h_3 e_3$$

where C is the normal of the cost plane thru R_0, determined by the three planes whose normals are e_1, e_2, and e_3 whose numbers numbers occur in the index of R_0.

We shall now show that the plane C does not cut the face

e_1 between the lines of intersection of planes e_1 , e_2 and e_1, e_3 which lines form face e_1.

The line of intersection of plane C with face e_1 is:

$$(5) \quad e_1 \times C = h_1 e_1 \times e_1 + h_2 e_1 \times e_2 + h_3 e_1 \times e_3$$

$$= h_2 g^{-e_3} - h_3 g^{-e_2}$$

We note that the last expression is the equation of a line which is outside the angle made by the lines

$$g^{-e_3} \text{ and } g^{-e_2}$$

which form face e_1. Note also that $e_1 \times e_3 = - e_3 \times e_1 = - g^{-e_2}$ and that is the reason for the minus sign in eq. (5) above. We also note that $e_1 \times e_1 = 0$ since g,s are column cofactors and in this case determinants with two rows the same. This is so in n dimensions. Equation (5) says that plane C does not cut face e_1 when the h_j are positive. Keep all e,s in cyclic order for all dimensions. This is a convenience. In the same way one can show that the cost plane C does not cut face e_2 nor e_3 and thus does not cut the polyhedron of cinstraint as long as the h_j are all positive.

For 4 dimensions we write:

$$(1) \quad C = h_1 e_1 + h_2 e_2 + h_3 e_3 + h_4 e_4$$

In 4 dimensions the plane C will cut any face e_j in three lines. For face e_1 one of these lines is :

$$(2) \quad (e_1 \times e_2 \times C) = h_1 (e_1 \times e_2 \times e_1) + h_2 (e_1 \times e_2 \times e_2)$$
$$+ h_3 (e_1 \times e_2 \times e_3) + h_4 (e_1 \times e_2 \times e_4)$$
$$= h_1 (0) + h_2 (0) + h_3 g^{-e_4} - h_4 g^{-e_3}$$
$$= h_3 g^{-e_4} - h_4 g^{-e_3}.$$

$$(e_1 \times e_2 \times e_4) = -(e_4 \times e_1 \times e_2) = - g^{-e_3}.$$

Equation (2) shows that one of the three lines of plane C cutting face e_1 passes thru the complement of two of the three lines

$$g^{-e_2} , \quad g^{-e_4} \quad \text{and} \quad g^{-e_3}$$

which form face e_1. A like result can be shown for the other two lines of plane C cutting face e_1. Those lines are $(e_1 \times e_3 \times c)$ and

($e_1 \times e_4 \times C$). A like result can be shown for faces e_2 , e_3 and e_4.

The Generalization and thus the

H Theorem

is immediate:

$$(1) \qquad C = h_1 e_1 + h_2 e_2 + h_3 e_3 + \ldots\ldots h_n e_n .$$

One of the (n - 1) lines of the hyperplane C cutting face e_1 is

$$(2) \qquad (e_1 \times e_2 \times e_3 \ldots e_{n-2} \times C) = h_1 (0) + h_2 (0) \ldots\ldots h_{n-1} g^{-e_n} - h_n g^{-e_{n-1}} = T.$$

Keep in mind the cyclic order of the vectors e_1, e_2 ... e_n, and that any product with a repeat factor is 0. A determinant with two equal rows is 0. Equation (2) above shows that the hyper-plane C does not cut face e_1 between the lines forming its face. A like result holds for the other lines of C cutting face e_1. A like result can be shown for the other faces e_2 ... e_n. Thus plane C does not cut the polyhedron of constraint as long as the h_i are positive.

The H Theorem

is a powerful tool for the complete solution of parametric linear programming.

The H Theorem

is a pioneering piece of work and we take a bit of pride in its fashioning.

Parametric linear programming is more general than ordinary linear programming. One can shape it to fit various pricing and scheduling to suit many requirements.

Polarization

and the

Theorem

It will be shown that the equations of constraint for a parametric system can be constructed als by the gamma theorem due to Polarization discussed in an earlier chapter in this book.

The H and theorems can serve as a chech on each other for each should give the same equations of constraint.

We now find the equations of constraint by the theorem of Polarization.

From solving the original problem we obtain the optimal vector R_0. We also write

$$C_0 = \frac{C_1}{R_1} + \frac{C_2}{R_2} + \frac{C_3}{R_3} + \dots \frac{C_n}{R_n}$$

where C_0 is the original cost vector and R_1, R_2 ... R_n are the nearest neighbors to R_0 on the original polyhedron of constraint. We now write our cost vector as:

$$C = C1 + K$$

$$K = 0 + K_1 + K_2 + \dots K_{n-1}.$$

$$(1) \qquad P = R_0 \cdot (C_1 + K).$$

Assuming a maximization in the original problem we replace R_0 by R_1, R_2, ... R_n and get by Polarization:

$$(1) \qquad R_1 \cdot (C_1 + K) = P$$

$$(2) \qquad R_2 \cdot (C_1 + K) = P$$

$$(n) \qquad R_n \cdot (C_1 + K) = P$$

Subtracting (1), (2), ... (n) from the (1) above we get

$$(n+1) \qquad (R_0 - R_1) \cdot (C_1 + K) = 0$$

$$(n+2) \qquad (R_0 - R_2) \cdot (C_1 + K) = 0$$

$$(2n) \qquad (R_0 - R_n) \cdot (C_1 + K) = 0.$$

We thus end up with n equations of constraint in n - 1 unknowns since K has n - 1 components.

For the three dimensional problem solved on pages 41 to 43 we obtained:

$$R_0 = 5 + 1 + 2$$

$$R_1 = 5 + 0 + 1.25$$

and if one had continued the table and found g^{-5} and g^{-6} the Other neighbors R_2 and R_3 would have been found.

$$R_2 = 4 + 6 + 6$$

$$R_3 = 3 + 4 + 3$$

Then we have

$$R_0 - R_1 = 0 + 1 + 0.75 = (0 + 4 + 3)/4$$
$$R_0 - R_2 = 1 - 5 - 4$$
$$R_0 - R_3 = 2 - 3 - 1 .$$ We can then writw:

$$(0 + 4 + 3) . (12 + K_1 + K_2) = 0 + 4 K_1 + 3 K_2 = 0$$
$$(1 - 5 - 4) . (12 + K_1 + K_2) = 12 - 5 K_1 - 4 K_2 = 0$$
$$(2 - 3 - 1) . (12 + K_1 + K_2) = 24 - 3 K_1 - 1 K_2 = 0$$

from which

$$5 K_1 + 4 K_2 = 12$$
$$3 K_1 + 1 K_2 = 24$$
$$4 K_1 + 3 K_2 = 0.$$

These last three equations are the same as those given by the

H Theorem

Some options are recorded on page 81 in section 2.

One can improve the derivation of the last set of constraint equations. The n neighboring points to R_0 may be written:

$$R_1 = R_0 + t_1 g^{-1}$$
$$R_2 = R_0 + t_t g^{-2}$$
$$R_n = R_0 + t_n g^{-n}.$$

Suppose we have altered our cost vector C_0 to C parametrically then we may write:

$$p_0 = C . R_0$$
$$p_1 = C . R_1 = C . (R_0 + t_1 g^{-1})$$
$$p_2 = C . R_2 = C . (R_0 + t_2 g^{-2})$$
$$p_n = C . R_n = C . (R_0 + t_n g^{-n})$$

Each p_j is smaller than p_0 since in max. problems R_j is on the near side of the cost plane thru R_0. Subtracting each p_j from p_o we get:

$$- C \cdot g^{-1} = 0$$
$$- C \cdot g^{-2} = 0$$
$$- C \cdot g^{-n} = 0;$$

the t_i all being positive were cancelled from the equations. In min. problems we only have to change signs in the last set getting:

$$C \cdot g^{-1} = 0$$
$$C \cdot g^{-2} = 0$$
$$C \cdot g^{-n} = 0.$$

These last two sets of constraint equations constitute the

Theorem

which is the result of Polarization.

The H and Γ Theorems give the same constraint system for the parametric objective function. On the following page are sketches for max. and min. Polarization.

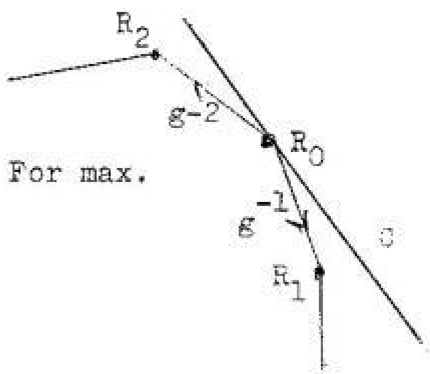


Fig. 24

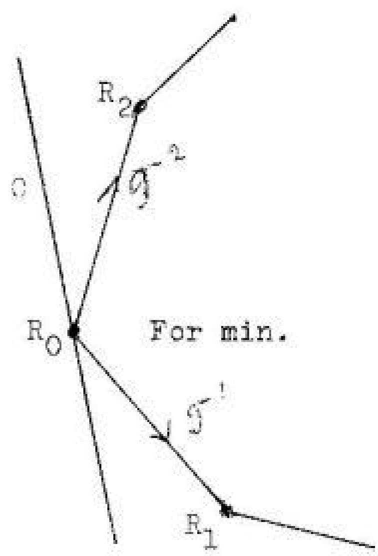


Fig. 25

All gammas leave R_0. The H and Γ theorems offer a complete solution for parametric linear programming problems of n dimensions. C teeters about R_0 according to parameters chosen in C_0.