Chapter 8.

1. The Transportation Problem

The Transportation Problem is of considerable interest to commercè and industry and much time and effort have been devoted to formulating and efficiently solving the related system of equations.

An inspection of the literature on the transportation problem seems to indicate that the most popular method of solution of the associated system of equations is an application of the simplex method or a variation of it. Many ingenious solutions, using conventional mathematics, have been devised to deal with the system of equations serving as constraints in the transportation problem.

Mutation Geometry does not need many theorems in order to deal with the transportation problem and its variants such as personell assignments, traveling salesman , etc.

The transportation problem is a linear programming problem in that it is formulated by a system of linear equations as constraints with an objective function to be minimized.

Formulation of the Equations of Constraint for the Transportation Problem.

A product is to be transported in amounts $a_1, a_2, \ldots a_m$, respectively from each of m shipping origins and received in amounts $b_1 \ldots b_n$ respectively by n shipping destinations. The cost of transporting a unit amount from the i th origin to the j th destination is C_{ij} and is assumed known for all combinations of i and j.

It is required to determine the amount x_{ij} to be transported over all routes (i, j) in order to minimize the total cost of the transportation.

For the developement of the equations of constraint we refer to the diagram below where x_{ij} is the amount transported from origin i to destination j. The total transported from origin i is a_i and the total received at destination j is b_j.

		Destinations						
	i \ j	1	2	3	j	..	n	
Origins	1	x_{11}	x_{12}	..	x_{1j}	..	x_{1n}	a_1
	2	x_{21}	x_{22}	..	x_{2j}	..	x_{2n}	a_2
	:	:	:	:	:	:	:	:
	i	x_{i1}	x_{i2}	:	x_{ij}	:	x_{in}	a_i
	.	.	.	.	.	.	.	:
	m	x_{m1}	x_{m2}	..	x_{mj}	..	x_{mn}	a_m
		b_1	b_2	..	b_j	..	b_n	

Fig. 26.

We define a vector r as:

$$(1) \qquad r = (x_{11}, x_{12}, \ \ldots \ x_{mn})$$

and a cost vector C as:

$$(2) \qquad C = (C_{11}, C_{12}, \ \ldots \ C_{mn})$$

then the total cost of the transportation is:

$$(3) \qquad P = C \cdot r$$

and we have the following relations from the table above:

$$(4) \qquad x_{ij} = a_i \ (i = 1, 2, \ \ldots \ m)$$

$$(5) \qquad x_{ij} = b_j \ (j = 1, 2, \ \ldots \ n)$$

$$(6) \qquad a_1 + a_2 + \ \ldots \ a_m = b_1 + b_2 + \ \ldots \ b_n$$

Equation (6) states that the sum sent is equal to the sum received.

From the table one sees that there are mn variables m + n equations of constraint represented in (4) and (5).

It may be shown that the m + n equations are redundant, having one more equations than required for the system to be independent, which means that one of the equations can be omitted or that one of the equations may be written as a linear combination of the others. In that case we have m + n - 1 independent equations. Any one of the equations may be omitted leaving an independent system. We illustrate the dependent system with a simple case of a 2 x 3 table and one may use the same scheme to show the truth of it for a higher number of variables. The 2 x 3 table and its equations are:

x_{11}	x_{12}	x_{13}	a_1
x_{21}	x_{22}	x_{23}	a_2
b_1	b_2	b_3	

$$(7) \qquad x_{11} + x_{12} + x_{13} = a_1$$

$$(8) \qquad x_{21} + x_{22} + x_{23} = a_2$$

$$(9) \qquad x_{11} + x_{21} = b_1$$

$$(10) \qquad x_{12} + x_{22} = b_2$$

$$(11) \qquad x_{13} + x_{23} = b_3$$

By actual adding and subtracting one obtains:

(7) = (9) + (10) + (11) - (8). In a similar way one can show a like result for m + n equations. There is always one redundant equation and so one equation is to be omitted in order for the system to be independent. One then has $m + n - 1$ independent wquations and mn variables. For $m = 2$, $n = 3$ we have

$$mn = 6 \quad \text{variables and}$$

$$m + n - 1 = 4 \quad \text{equations.}$$

In order to solve a system of equations one must have as many equations as unknowns. Most high school students know this and it needs no proof. One might give a Heaviside proof: (" it works "). We borrow two equations from the set of inequalities

$$x_{ij} = 0$$

and then we have six equations and six unknowns. It makes only a slight difference as to which two we choose. When the cost vector C is numerically one will see to make the choice advantageously so that only a few iterations are required to obtain the proper r which will minimize the objective function P.

When one solves the six equations in six unknowns one gets a point represented by a vector r with six components some of which may be negative. Vectors with negative components are unaceptable and have to be corrected till all components are positive. It will be shown later how to do this.

The system of constraints represents a hyper-convex polyhedron . The vertices of this polyhedron are of prime importance to us. One of them gives the objective function its minimum value. We have to find that particular vertex.

2. Polarizing the Plane and Space; Analogy and Generalization.

If a point lies on a given line its equation may be written:

(1) $$a \cdot r = b$$

where b is a positive scalar and a is a vector whose direction is along the normal to the given line and r is a vector from the origin to any point of the given line.

If any point is on the far side of the given line from the origin one may write the inequality:

(2) $$a \cdot r \geq b.$$

If the point is on the near side of the line to the origin one may write the inequality:

(3) $$a \cdot r \leq b.$$

A similar statement is true about the plane polarizing space.

A hyper plane polarizes a hyper space.

The notions of polarization are very usful in linear programming in the styling of the New Science of Mutation Geometry.

In high school geometry one learns that two planes intersect in a straight line. One can say this in another way: two planes in 2 dimensions intersect in a straight line in a space of 3 dimensions, As a generalization of this we say that n-1 planes in n-1 dimensions intersect in a straight line in a space of n dimensions.

3. (Gamma) Vectors

We have called the line of intersection of n - 1 hyper planes in a space of n dimensions a(gamma) Vector. Notice that the intersection of two ordinary planes is a particular case of a vector.

The Vector is a construct of the New Science of Mutation Geom.

The solution of n equations in n unknowns gives a point; stated differently n hyperplanes in n unknowns determine a point.

Let

(1)	$a \cdot r = A$
(2)	$b \cdot r = B$
(3)	$c \cdot r = C$
(4)	$d \cdot r = D$
(n)	$\cdot r =$

be the equations of n hyperplanes in n unknowns where a, b, ... are their normals. One can compute n ftom the list by omitting one equation in tern from the list. They are:
where the exponents indicate the equation omitted. This means that n gammas leave each point in n dimensional space. This is the CARDINAL PRINCIPLE in linear programming in the styling of the New Science of Mutation Geometry.

Take note that:

$$a \cdot \gamma^{-2} = a \cdot \gamma^{-3} = a \cdot \gamma^{-4} = \dots a \cdot \gamma^{-n} = 0.$$

since each product is a determinant with two rows the same. A like result can be written for each of the other gamma. The gamma vectors play a significant role in the new foundation of linear programming in the new styling of Mutation Geometry.

The gamma vectors short circuit the older simplex formulation of linear programming with its vexations of slack variables, bases, expansions, degeneracy, cyccling, etc; a slight bit of baggage to cast off.

The γ vectors are the edges of the faces of the hyper-polyhedrons of the constraint system. We have devised a number of ways to efficiently compute them for any system of constraints.

For an abstract of a paper on the New Science of Mutation Geometry, presented before the Ohio Section of the Mathematical Assoociation of America, meeting at Miami University in Oxford, Ohio; see the Monthly for Aug - Sept. 1959, page 645.

If we solve any n of the equations of constraint in n unknowns we shall get a point. Let r_0 be the vector representing this point. If all the components of this vector are positive and the vector satisfies the whole system of constraints we shall call the point so obtained a hull point. The point represents a vertex of the polyhedron of constraint. If the vector has all its components positive but does not satisfy all the equations of constraint of if it has one or more negative components we shall call the point an ahull point. In that case it does not represent a vertex of the polyhedron of constraint. By means of our gamma vectors one may go from a hull point to a hull point. We shall show several ways of getting on a vertex of the polyhedron. Once on the hull the gamma vectors will sample the other vertices for the optimum one of them.

Let us now suppose that we have, by some means, found a a vector r_0 which satisfies the system in diagram 26. We can then get another solution:

$$r = r_0 + t\gamma$$

where t is a positive scalar and gamma(γ) is an orthovector of n - 1 of the normals of the planes determining the point r_0. Put the last equation into the system of constraints in table 26 and getthe table of gamma(γ) constraints :

		Destinations						
	$i \backslash j$	1	2	3 ..	J	- -	n	
	1	γ_{11}	γ_{12}	γ_{13} --	γ_{1J}	- -	γ_{1n}	0
	2	γ_{21}	γ_{22}	γ_{23} --	γ_{2J}	- -	γ_{2n}	0
Origins	:	- -	- -	- -	- -	- -	- -	0
	i	γ_{i1}	γ_{i2}	γ_{i3} --	γ_{iJ}	- -	γ_{in}	0
	:	- -	- -	- -	- -	- -	- -	0
	m	γ_{m1}	γ_{m2}	γ_{m3} -	γ_{mJ}	- -	γ_{mn}	0
		0	0	0	0	0	0	

Fig. 27.

We have devised a number of ways of computing the . One easy way is to solve n - 1 of the gamma( ) equations in the gamma table for the various gamma( ) components in terms of a chosen component then one will get some such relation as:

$$= \quad (1 + \ldots 0 + \ldots - 1 \ldots)$$

where is the chosen component. The sign of the is at the disposal of the one doing the computing. Another way is to take the column cofactors of n - 1 of the equations of constraint which give the solution under consideration. Another way is by inspection of a gamma( ) table. We shall illustrate some of these schemes in the working of illustrative problems.

Problem. Minimize the objective

$$P = c \cdot r$$

where c is given by the table

$$c = \begin{matrix} c_{11} + c_{12} + c_{13} \\ c_{21} + c_{22} + c_{23} \end{matrix} = \begin{matrix} 2 + 3 + 1 \\ 4 + 1 + 5 \end{matrix}$$

and r is given by the table

$$r = \begin{matrix} x_{11} + & x_{12} + & x_{13} & 10 \\ x_{21} + & x_{22} + & x_{23} & 14 \\ 7 & 9 & 8 & \end{matrix}$$

$$= \begin{matrix} \gamma_{11} + & \gamma_{12} + & \gamma_{13} & 0 \\ \gamma_{21} + & \gamma_{22} + & \gamma_{23} & 0 \\ 0 & 0 & 0 & \end{matrix}$$

$$\gamma = \gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{21} + \gamma_{22} + \gamma_{23}$$

From the gamma table we write the equations:

$$\gamma_{21} + \gamma_{22} + \gamma_{23} = 0$$
$$\gamma_{11} + \gamma_{21} = 0$$
$$\gamma_{12} + \gamma_{22} = 0$$
$$\gamma_{13} + \gamma_{23} = 0$$

Setting gamma $\gamma_{23} = 0$ and solving the remaining ones in terms of γ_{21} (chosen arbitrarly) one gets:

$\gamma_{11} = -\gamma_{21}$

$\gamma_{22} = -\gamma_{21}$

$\gamma_{12} = \gamma_{21}$

$\gamma_{13} = 0$, Chosen

$$\gamma = \gamma_{21}(-1+1+0+1-1+0)$$

We arbitrarily choose $x_{22} = x_{23} = 0$ and solve the four remaining equations and get:

$r_0 = -7 + 9 + 8 + 14 + 0 + 0.$

The sign of the gamma is at our disposal. We write it as:

$$\gamma = 1-1+0-1+1+0$$

We multiply γ by t and add it to r_0 and get

$r_1 = r_0 + t\gamma = (t-7) + (9-t) + 8 + (14-t) + t + 0.$

whence $P = C \cdot r = 77 - 4t.$

We want P to be as small as possible consistent with r having no negative components. One sees that t has to be 9 and so

$P = 77 - 4(9)\ 77 - 36 = 41 = \text{Min.}$

$r_1 = 2 + 0 + 8 + 5 + 9 + 0.$

Writing r_1 in tabular form:

$$r_1 = \begin{array}{cccc} 2 & 0 & 8 & 10 \\ 5 & 9 & 0 & 14 \\ 7 & 9 & 8 & \end{array}, \quad \gamma^{-1} = \begin{array}{cccc} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{array}, \quad \gamma^{-2} = \begin{array}{cccc} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \end{array}$$

From the first gamma we get:

$r_2 = r_1 + t\gamma = (2-t) + t + 8 + (5+t) + (9-t) + 0.$

$P_2 = C \cdot r_2 = 41 + 4t.$

$t = 0$, then $r_2 = r_1$ and $P_2 = 41.$

From the second gamma we get $r_3 = r_1$ and $P_3 = 41.$

From the second gamma we get $r_3 = r_1$ and $P_3 = 41$. Thus all the neighbors of r_1 give a larger or equal value of P than than that of r_1.

$P_1 = C \cdot r_1 = 41 = \text{Min.}$

$P_2 = C \cdot r_2 = 41$

$P_3 = C \cdot r_3 = 41.$

For the calculation of gamma one, γ^{1}, we look at the zeroes in r_1 and for the first 0 we write 1 and fill in the other spaces in the gamma table with a o corresponding to the 0 in the r_1 table. Then one calculates the other numbers in the gamma table in agreement with the constraints on its edges. For the gamma, γ^{2}, one simply interchanges the 1 and 0 and proceeds as in the first case. One does not have to do much calculating. A little observation shows that one gets the min vector by adding to r_0 the product of the gamma by the smallest positive component of r_0 which lies opposite -1 in the gamma.

We now solve a problem with 12 unknowns. Given the cost matrix

$$(1) \qquad C = \begin{matrix} 1 + & 2 & + & 3 & + & 4 \\ 4 + & 3 & + & 2 & + & 0 \\ 0 + & 2 & + & 2 & + & 1 \end{matrix} =$$

Find the min. of

$$P = C \cdot r$$

$$r = (x_1 + x_2 + \cdots \; x_{34})$$

under the constraint system:

$$(2) \qquad \begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & 6 \\ x_{21} & x_{22} & x_{23} & x_{24} & 9 \\ x_{31} & x_{32} & x_{33} & x_{34} & 10 \\ 4 & 6 & 8 & 5 & \end{matrix} \qquad x_{ij} \geqq 0$$

In (2) we have 6 equations and 12 unknowns $x_{11} \ldots x_{34}$.

We now must choose 6 (3 + 4 − 1) out of the 12 unknowns to be 0 or we could choose 6 equations from the 12 inequalities x_{ij} to go with the six equations making 12 equations in 12 unknowns which can be solved.

For the construction of our gamma vectors γ we shall think in terms of mn + 1 equations in mn unknowns.

There will be 6 gammas γ for each r solution. The number of gammas is the same as the number of zeroes chosen. 6 of the equations remain fixed for all gammas γ . The gamma vectors play a significant role in all linear programming from the Mutation Geom. Viewpoint.

Making $x_{13} = x_{14} = x_{21} = x_{22} = x_{24} = x_{32} = 0$ we get the table table of values:

$$(3) \qquad r_0 = \begin{array}{ccccc} 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 8 & 0 & 8 \\ 4 & 0 & 0 & 6 & 10 \\ 4 & 6 & 8 & 6 & \end{array} \qquad P = C \cdot r = 34.$$

This table of values is degenerate in that it has 8 zeroes when only 6 are needed. Looking at the table of C values we see that the smallest values corresponding to 0,s is C_{11} and C_{33}. We put a cross inside these 0,s and call them floating zeroes since one can construct gamma vectors only for mn - 1 in mn unknowns. One can write the gamma table at sight.

If any γ has a negative component corresponding to a zero in the components of r_0 that γ is said to be inactive and is discaded for it would give negative components in the resulting r.

We note also that when one of the equations of inequality is omitted it will have a coefficient of 1 before it as $1\, x_{ij} \geq 0$ and the resulting γ must make a positive angle $\leq 90°$ with this vector from which it is departing. One does not depart from from floating vectors.

Each γ row and column will have either all o,s or + 1, or - 1. One needs to write down only the + 1 and the - 1 and no zeroes. We note also that

$$r_1 = r_0 + t\,\gamma$$

$$P_1 = C \cdot r_1 = P_0 + t\, C \cdot \gamma$$

Since we want P_1 to be as small or smaller than P_0 ther is no point in dealing with γs whose product $C \cdot \gamma$ is +. Use only those whose product is 0 or negative. From the r_0 table we compute the following γs

$\gamma^{-13} =$

-1		1		0
				0
1		-1		0
0	0	0	0	

inactive

$\gamma^{-14} =$

-1			1	0
				0
1			-1	0
0	0	0	0	

inactive

$\gamma^{-21} =$

				0
1		-1		0
-1		1		0
0	0	0	0	

$C . \gamma^{-21} = +$

$\gamma^{-22} =$

1	-1			0
	1	-1		0
-1		1		0
0	0	0	0	

$C . \gamma^{-22} = +$

$\gamma^{-24} =$

				0
		-1	1	0
		1	-1	0
0	0	0	0	

$C . \gamma^{-24} = -$

$\gamma^{-31} =$

1	-1			0
				0
-1	1			0
0	0	0	0	

$C . \gamma^{-31} = +$

We see from the γ_λ that γ^{-24} is the only one we need to deal with. We see from the table of r_0 that 6 is the smallest component opposite which a negative component of γ^{-24} falls. We multiply γ^{-24} by 6 and add to r_0:

$$r = r_0 + 6\gamma^{-24}$$

$$C . r = C . r_0 + 6 C . \gamma^{-24}$$

$$P = P_0 + 6 (- 1)$$

$$P = 34 - 6 = 28 = \text{min.}$$

$r =$

0	6	0	0	6
0	0	2	6	8
4	0	6	0	10
4	6	8	6	

The answer is a degenerate minimum with 7 zeroes where we only need 6 in the solution. It satisfies all the boundary conditions.

In calculating the gammas it is always advamtageous to fill in the zeroes first . For example, if any row or column has all zeroes but one in r_0 then in the gamma that row or column has all zeroes. One can then easily see how to place the + 1 and the - 1. Heed this for it speeds the derivation of the gammas.

4 A Useful Idea

In the transportation problem it is convenient to write down a first solution at sight. If we have m shipping origins and n destinations the number of unknowns x_{ij} is mn. The number of independent equations is $m + n - 1$. The no of zeroes to be chosen is U:

$$U = mn - (m + n - 1) = (m - 1)(n - 1)$$

It is a very important and useful expression. It means that we can put zeroes in the first m - 1 rows up to the nth column and b_1, b_2, b_3 ... b_{n-1} .. x_{mn} in the last row or mth row. The last column will have a_1, a_2, ... a_{m-1} , x_{mn} where

$$x_{mn} = b_n - (a_1 + a_2 + .. a_{m-1})$$

$$= a_m - (b_1 + b_2 + .. b_{n-1}).$$

and the resulting expression is a first solution. It may have 1 negative component but we have shown how to deal with such solutions with a gamma attack. Other interpretations can be given to the U expression above. Some give all pesitive component feasible initial solutions. Some examples follow. According to the U formula the solution of the problem

x_{11}	x_{12}	x_{13}	a_1
x_{21}	x_{22}	x_{23}	a_2
b_1	b_2	b_3	

is r_0 =

0	0	a_1	a_1
b_1	b_2	$b_3 - a_1$	a_2
b_1	b_2	b_3	

Herw m = 2, n = 3, m - 1 = 1, n - 1 = 2

This means 1 row with two zeroes in it. See the diagram above.

r_0 =

0	0	10	10
7	9	-2	14
7	9	[illegible]	

γ^{-11}

1	0	-1	0
-1	0	1	0
0	0	0	

γ^{-12}

0	1	-1	0
0	-1	1	0
0	0	0	

One can multiply γ^{-11} by any positive number from 2 to 7 and add it to r_0 and get feasible solutions. Multiply γ^{-11} by t and add to r_0 and get:

$$r = r_0 + t\gamma^{-11} = t + 0 + (10 - t) + (7 - t)$$
$$+ 9 + (t - 2).$$

$$P = C \cdot r = 2t + 37.$$

Our objective function will be the smallest when t is the smallest consistent with r having all positive components . Here $t = 2$ and

$$P = 4 + 37 = 41 = \text{Minimum}.$$

One can also get feasible solutions by multiplying γ^{-12} by positive numbers from 2 t0 9 inclusive. This is left for the students. It will be fun to watch the march of the gammas. Have fun. There are exciting discoveries to be made. Watch for them. Once having a feasible solution one can then be off to the races with the march of the gammas.

We now reintreprete the U formula for the number of zeroes to be chosen.

$$U = (m - 1)(n - 1).$$

Zeroes may be shifted from the $(m - 1)$ row to the m th row by replacing them by the corresponding b_j. We do an illustrative example.

b_1		$a_1 - b_1$	a_1
	b_2	$a_2 - b_2$	a_2
b_1	b_2	b_3	

r_0 =

7		3	10
	9	5	14
7	9	8	

r_0 is a feasible solution with all positive components. One could also get another feasible solution by interchanging rows with b_1 and b_2 . This is left for the students. Some solutions of this type give negative components in the resulting r vector but this does not matter in the end because the gammas dispose of them as we have shown.

If the negative component was large one might not be able to correct the solution with one γ addition. We do an example:

r_0 =

				1	1
				5	5
3	3	3	2	- 4	7
3	3	3	2	2	

γ^{-23}

		1		-1	
		- 1		1	

Add $3\gamma^{-23}$ and get

r_1 =

				1	1
		3		2	5
3	3	0	2	- 1	7
3	3	3	2	2	

$$r_1 = r_0 + 3\gamma^{-23}$$

$\gamma^{-14} =$

			1	-1	
			-1	1	

$r_2 =$

			1	0	1
		3		2	5
3	3	0	1	0	7
3	3	3	2	2	

$r_2 = r_1 + 1\,\gamma^{-14}$

r_2 is an all positive component feasible solution . It satisfies the boundary conditions. One notices that the choices of our gammas can materially hasten the elimination of a negative component. In the last scheme there can never be more than 1 negative component. There are naturally times when there are no negative components, in the initial solution. For example:

$r_0 =$

				1	1
				3	3
3	1	1	2	(2)	9
3	1	1	2	6	

$y_{35} = b_5 - (a + a_2) = (2)$

r_0 is an all positive component initial feasible solution. One can soon write down the 8 gammas and with a given cost matrix test whether it is the min. vector.

The last scheme for finding an initial solution does not depend on the values of the C_{ij} in the cost matrix and for that reason the solution may be far from the min. solution and thus would require more computation than for a solution which was initially near the min. solution.

5. Material Allocations

That any scheme for calculation an initial solution, which does not depend on the C_{ij}, would produce a solution near the min., is improbable. It is a matter of odds: onlu a few solutions near the min and more not so near. We shall allocate the material by rows.

Let C_{1j} be the minimum element in the first row. We set $x_{1j} = a_1$ if $a_1 = b_j$ or $x_{1j} = b_j$ if $a_1 = b_j$. In the first case we have allocated all the material in row one and we change a_1 to 0 and replace b_j by $b_j - a_1$. We then go to the second row and repeat the process. In the second case we replace a_1 by $a_1 - b_j$ and replace b_j by 0. Using this scheme we find an initial solution for the following problem.

C =

1	1	2	1	3	
2	2	1	2	1	
4	3	1	2	2	

r_0 =

1	0	0	0	0	1
0	0	1	0	2	3
2	1	0	2	4	9
3	1	1	2	6	

$P = C \cdot r_0 = 27$

We now write down the 8 gammas from r_0. To avoid drawing new diagrams we write the coordinates where the 1 and − 1 are located. Then

		$C \cdot \gamma$
$\gamma^{-12} =$	12 − 11 + 31 − 32	= 1
$\gamma^{-13} =$	13 − 23 − 11 + 21	= 2
$\gamma^{-14} =$	14 − 34 + 31 − 11	= 2
$\gamma^{-15} =$	15 − 11 + 31 − 35	= 4
$\gamma^{-21} =$	21 − 11 − 31 + 13	= 2
$\gamma^{-22} =$	22 − 32 − 25 + 35	= 0
$\gamma^{-24} =$	24 − 34 − 25 + 35	= 1
$\gamma^{-33} =$	33 − 23 − 35 + 25	= − 1

γ^{-33} is the only useful gamma in the 8 computed gammas for r_o. Then

$$r_1 = r_o + \gamma^{-33}$$

$$P_1 = C \cdot r_1 = C \cdot r_o + C \cdot \gamma^{-33} = 27 - 1 = 26. = \text{Min.}$$

The reason 26 is the minimum is that all the gammas of r_1 are either inactive (give negative components in the resulting r) or give positive products with C. The tabular form for r_1 is written below.

r_1 =

1	0	0	0	0	1
				3	3
2	1	1	2	3	9
3	1	1	2	6	

As a model, we now show in detail how to compute the γ^{-33} from the r_o table. We first replace the 0 in the 33 square by a 1 then omit all other numbers in the table but the 0 s. one then replaces the red boundary numbers by red zeroes. One then fills in the table with either 1,s, 0,s or - 1,s to fit the boundary conditions. One can do it at sight. The skeleton table below is the starting form. The second table is the table filled in. It is the required γ^{-33}. All gamma,s are calculated in this manner for the present.

	0	0	0	0	0
0	0		0		0
		1			0
0	0	0	0	0	

Skeletal γ^{-33}

0	0	0	0	0	0
0	0	-1	0	1	0
0	0	1	0	-1	0
0	0	0	0	0	

Filled in γ^{-33}

One can calculate gamma vectors from any point vector r in the same way. One can calculate 8 of them from the new r_1. It will be interesting practice for the student and then to find new vectors from the old vectors. Note that $r_1 = r_o + \gamma^{-33}$. By means of the gammas one can find all the neighbors (vertices) of a given r (vertex. Then one can find their P = C. r values and choose the best of them and repeat the process. This is Linear Programming in the styling of the New Science of Mutation Geometry. For a presentation of the New Science before the American Mathematical Assn. meeting at Miami University in Oxford, Ohio ; see the Monthly for Aug- Sept page 645 (1959). The initial value r_o was only one iteration away from the min. vector r_1. The allocation scheme for finding an initial solution can be operated swiftly and efficiently. One only needs a little practice. The gammas will do the rest.

We have devised several schemes for computing gamma vectors. For completeness we derive γ^{-12} for the r_1 vector above in a linear fashion. Most of the work is in making the diagram. The other 7 gammas can be done in the same way. In practice I never do them that way when they can be done at sight from a tabular form. See below for the computation

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{31}	X_{32}	X_{33}	X_{34}	X_{35}		
1		1														$X_{11}=$	0
2			1													$X_{12}=$	0
3				1												$X_{13}=$	0
4					1											$X_{14}=$	0
5						1										$X_{15}=$	0
6							1									$X_{21}=$	0
7									1							$X_{22}=$	0
8													1			$X_{23}=$	0
9						1	1	1	1	1						$X_{24}=$	3
10											1	1	1	1	1	$X_{25}=$	9
11	1					1					1					$X_{31}=$	3
12		1					1					1				$X_{32}=$	1
13			1					1					1			$X_{33}=$	1
14				1					1					1		$X_{34}=$	2
15					1					1					1	$X_{35}=$	6
γ^{-12}	-1	1	0	0	0	0	0	0	0	0	1	-1	0	0	0		

Keep in mind that the γ^{-12} is to be perpendicular to each row from 2 to 15 and have 1 in column 2 for the gamma vector must make an acute angle with the normal of the hyperplane from which it is leaving. The gamma with a 0 in each component corresponding to each 1 in lines 2 to 8 satisfies the perpendicular requirement for lines 2 to 8. One then goes one then goes to line 12 which gives the - 1 in the 12 th component of γ^{-12}. We get the 0 in the 8th component of γ^{-12} ffom row 13. We get the 0 in the 10th component from row 9. We get the 0 in the 15th component from row 15. Finally we get the - 1 in the 1st component from row 11. This completes γ^{-12}. It is the same as that computed from the tabular form but not so compact and not so convenient. The other 7 gammas can be computed in like fashion. The main work here was in the construction of the diagram. One could also compute the gamma above by computing the column cofactors of the remaining 14 equations. Each component of the gamma would be a 14 by 14 determinant and who wants to be punished like that when they can be written down at sight.

The row allocations for an initial solution followed by a gamma attack for a minimum may entail the least amount of work and that is desirable.

O	1	2...	j...	n	B	1	2...	j.....n	
1	- 1	0	0	0	0				
2	0	- 1	0	0	0				
j	0	0	- 1	0	0				
n	0	0	0	- 1	0				
m+1	a1	a2	aj	an	B	b1	b2	bj	bn
m+2	a1	a2	aj	an	B	b1	b2	bj	bn
i	a1	a2	aj	an	B	b1	b2	bj	bn
m	a1	a2	aj	an	B	b1	b2	bj	bn
R						xy1	xz2	xij	xhn
C	c1	c2	cj	cn		c1	c2	cj	cn
S						S1	S2	Sj	Sn
Ri	0	0	xij	0	S	(			)
rp	d1	d2	dj	dn					
rq	d1	d2	dj	dn					
rn	d1	d2	dj	dn					
j	c.rp	ep	eq	en	f	g	tp	tq	tn
m+1	c.rq	ep	eq	en	f	g	tp	tq	tn
m+2	c.rn	ep	eq	en	f	g	tp	tq	tn
m		ep	eq	en	f	g	tp	tq	tn
T							tup	tvq	twn
m+1	f1	f2	fj	fn	=	tup rp			
m+2	g1	g2	gj	gn	=	tvq rq			
m+3	h1	h2	hj	hn	=	twn rn			
Ru	k1	k2	kj	kn	S				
Rv	k1	k2	kj	kn	S	(			)
Rw	k1	k2	kj	kn	S				

Table above designed by Dr. Beckham Martin, Feb. 12, 1969