December 7, 1984

Non-Hull Solution

Given a linear programming problem:

$$P = C * R$$

subject to:

$$A * X < B$$

a general solution of the objective function

$$P = (c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_N) * R$$

is given by:

$$\begin{aligned} R = {} & (P / c_1 - t_2 - t_3 \ldots - t_N) + \\ & (c_1 / c_2) * t_2 + \\ & (c_1 / c_3) * t_3 + \ldots \\ & (c_1 / c_N) * t_N \end{aligned}$$

Where t is a parameter to be determined.

If R is put into P it satisfies giving an identity. The R equation helps one to put the objective function P into the constraint system, thus providing more solution leverage. We illustrate with some numerical examples.

$$\begin{aligned} \text{Max } P &= C * R \\ &= (12 \quad 1 \quad 1) * R \end{aligned}$$

subject to:

(4) $(1 \quad 4 \quad -3) * R < 10$

(5) $(5 \quad 6 \quad -8) * R < 15$

(6) $(1 \quad -3 \quad 4) * R < 10$

(7) $(1 \quad 1 \quad -1) * R < 4$

Our R solution then becomes:

$$R = (P/12 - t_2 - t_3) + 12*t_2 + 12*t_3$$

Put R into (4), (5), (6), and (7) and get:

$(4)'$ $\quad P/12 + 47*t_2 - 37*t_3 < 10$

$(5)'$ $\quad 5P/12 + 67*t_2 - 101*t_3 < 15$

$(6)'$ $\quad P/12 - 37*t_2 + 47*t_3 < 10$

$(7)'$ $\quad P/12 + 11*t_2 - 13*t_3 < 4$

Eliminating t_2 from $(6)'$ and $(7)'$ we get

(8) $\quad P < 64.5 - 9*t_3$

Eliminating t_2 from $(5)'$ and $(6)'$ we get:

(9) $\quad 3*P < 175 + 84*t_3$

Eliminating t_3 from (8) and (9) we get:

$$111*P < 6993$$

therefore

$$P = 6993/111 = 63 \quad (\text{Maximum})$$

which is the answer given in the GAMMA solution.

Put P into (8) and we get:

$$t_3 = 1/6 = 2/12$$

put P and t_3 into $(7)'$ and get

$$t_2 = 1/12.$$

Put P, t_2 , and t_3 into R and get

$$R = (5 \quad 1 \quad 2)$$

which is the same as the GAMMA solution:

$$P = (12 \quad 1 \quad 1) * (5 \quad 1 \quad 2)$$

$$= 60 + 1 + 2$$

$$= 63 \quad (\text{Maximum})$$

which confirms each answer. Notice that no polyhedron was needed in the solution technique.

Oct 15, 1984

Non-Hull Solution

(ALSO INTEGER SOLUTION)

Minimize

$$P = C * R$$

$$= (10 \quad 14 \quad 21) * R$$

subject to:

(4) $(8 \quad 11 \quad 9) * R > 12$ (redundant)

(5) $(2 \quad 2 \quad 7) * R > 14$

(6) $(9 \quad 6 \quad 3) * R > 10$

A generalized solution for the P equation is:

(7) $$R = (P/10 - t_2 - t_3) + (10/14) * t_2 + (10/21) * t_3$$

valid for all values of t_2 and t_3. Put (7) into (5) and (6) and get:

(8) $(P/5) - (4/7) * t_2 + (4/3) * t_3 > 14$

(9) $(9 * P/10) - (33/7) * t_2 - (53/7) * t_3 > 10$

Setting $t_2 = 0$ and eliminating t_3 from the resulting equations, we get:

$$P = 2506/57 = 43.9$$

From (8)

$$t_3 = 3.9$$

and (7) becomes

$$R = (0.49 \quad 0 \quad 1.86)$$

A GAMMA solution gives;

$$R = (28 \quad 0 \quad 106)/57$$

which reduces to the same answer.

To get an integer solution from (7) satisfying (8) and (9) we note that if we set $t_3 = 21/10 = 2.1$, the last term of (7) becomes 1 and the first term of (7) becomes;

$$(P/10 - 0 - t_3) = P/10 - 2.1$$

$$= 0, 1, 2, 3 \ldots$$

Depending on whether we make

$$P = 21, 31, 41, 51 \ldots$$

The R value becomes

$$R = (0 \quad 0 \quad 1)$$

$$R = (1 \quad 0 \quad 1)$$

$$R = (2 \quad 0 \quad 1)$$

$$R = (3 \quad 0 \quad 1)$$

none of which satisfies (5) or (6). We next set

$$t_3 = 2(21/10)$$

$$= 4.2$$

then

$$(P/10 + 0 - 4.2) = (52/10 + 0 - 4.2)$$

$$= 5.2 - 4.2$$

$$= 1$$

Then

$$R = (1 \quad 0 \quad 2)$$

Set (5) and (6) feasibly, and

$$P = 52$$

is the integer minimum.

Note that in this solution no use of a polyhedron is made in either the non-integer or integer solution. Equation (7), unheard of in the literature, is a powerful tool in linear programming. I write the general expression: The general solution of

$$P = C * R$$

$$= (c_1 \quad c_2 \quad c_3 \quad \ldots c_N) * R$$

is:

$$R = (P / c_1 - t_2 - t_3 - \ldots t_N) + (c_1 / c_2) * t_2 + (c_1 / c_3) * t_3 + \ldots (c_1 / c_N) * t_N$$

where t is a parameter to be determined.

Example:

Maximize (integer):

$$P = C * R$$

$$P = (3 \quad 4 \quad 4) * R$$

$$R = (p/3 - t_2 - t_3) + (3/5) * t_2 + (3/4) * t_3$$

set

$$t_2 = (5/3) * y_2$$

$$t_3 = (4/3) * y_3$$

then

$$R = (P - 5 * y_2 - 4 * y_3)/3 + y_2 + y_3$$

We also have the constraint equations

(4) $(2 \quad 3 \quad 0) * R < 8$

(5) $(0 \quad 2 \quad 5) * R < 10$

(6) $(3 \quad 2 \quad 4) * R < 16$

Put R into (4) - (6) and get:

(7) $2P - y_2 - 8y_3 < 24$

(8) $2y_2 + 5y_3 < 10$

(9) $P - 3y_2 < 16$

whence

$$82P = 1552$$

therefore

$P = 18.93$ (maximum)

For integers we may write:

$P = (\ 3 \quad 5 \quad 4\ ) * R = 18 - h$

$R = (\ 0 \quad (\ 2 - h\ ) \quad (\ 2 + h\ )\ )$

$G = (\ 4 \quad 0 \quad -3\ )$

$R = R_0 + t * G$

$= (\ 4 * t \quad (\ 2 - h\ ) \quad (\ 2 + h - 3 * t\ )\ )$

where t and h are parameters.

Set $t = 1$, $h = 2$, and get:

$R = (\ 4 \quad 0 \quad 1\ )$

$P = 18 - h$

$= 18 - 2$

$= 16$ (integer maximum)

$P = C * R$

$= (\ 3 \quad 5 \quad 4\ ) * (\ 4 \quad 0 \quad 1)$

$= 12 + 0 + 4$

$= 16$

R satisfies the P equation for all values of h.

$R = (\ 4 \quad 0 \quad 1)$

satisfies (4) and (6) exactly and (5) feasibly.

June 11, 1985

Non-Hull Solution

Maximize:

$$P = C * R$$

$$= (\ 3 \quad 4\ ) * R$$

subject to:

(3) $(\ 1 \quad -2\ ) * R < 3$

(4) $(\ 1 \quad 1\ ) * R < 9$

(5) $(-3 \quad 1\ ) * R < 1$

(6) $(\ 1 \quad 2\ ) * R < 14$

(7) $(\ 2 \quad -1\ ) * R < 9$

(8) $(-2 \quad 1\ ) * R < 2$

(9) $(\ 3 \quad 1\ ) * R < -14$

(10) $(\ 5 \quad -2\ ) * R < -16$

A general solution of the P equation is:

$$R = (\ P/3 - t_2\ ) + (\ 3/4\ ) * t_2$$

Put R into (3) - (10) and get:

(3)' $P/3 - (\ 15/2\ ) * t_2 < 3$

(4)' $P/3 - (\ 1/4\ ) * t_2 < 9$

(5)' $-P - (\ 15/4\ ) * t_2 < 1$

(6)' $P/3 + (\ 1/2\ ) * t_2 < 14$

(7)' $2P/3 - (\ 11/4\ ) * t_2 < 9$

(8)' $-2P/3 + (\ 11/4\ ) * t_2 < 2$

(9)' $-P + (\ 9/4\ ) * t_2 < -14$

(10)' $-5P/3 + (\ 13/2\ ) * t_2 < -16$

Eliminate t_2 from (3)', (4)', (7)' by means of (9)' and get three P values:

$P_3^9 > 14.56$

$P_4^9 < 33.5$ (Its R does not satisfy all constraints.)

$P_7^9 > 14.6$

New eliminate t_2 from (3)', (4)', (7)' by means of (10)' and get three P values:

$P_3^{10} > 14.36$

$P_4^{10} < 31.14$ = Maximum (same as GAMMA solution)

$P_7^{10} > -58$

P_4^9 and P_4^{10} are the only ones of the six values that have a real meaning. One can now evaluate the components of the symbolic vector R. One can do it in two ways: put P = 31.14 into (10)' giving $t_2 = 5.52$, or put P and t_2 into R and get;

R = (4.86 4.14) (as a check)

P = C * R

= (3 4) * (4.86 4.14)

= 31.14.

Again since we know that P comes from (4) and (10) we can solve these two and get:

R = (4.86 4.14)

R satisfies (10) and (4) exactly and the others feasibly. It is good to have a check on the GAMMA and non-hull solutions and vice versa. Notice how the determination of the maximum in the non-hull solution comes about. It teaches us a lot.

The symbolic vector was a master stroke, enabling us to put the objective function into the constraint system. It eliminated the computation of the vertices on the polyhedron of constraints. It offers a new push in linear programming. It is in its infancy and has a vast field for exploration.

November 14, 1984

Non-Hull Solution

(INTEGER SOLUTION ALSO)

The first page of solutions is a GAMMA solution of the problem. Minimize the objective function:

P = C * R

= (10 14 21) * R

subject to:

(4) (2 2 7) * R > 14

(5) (9 6 3) * R > 10

from which we get:

(6) R = (28 0 106)/57

(7) P = (10 14 21) * (28 0 106)/57

= 44

To get an integer solution for this problem, we write the expression:

(8) P = (10 14 21) * R = (44 + n)

(9) G = (21 0 -10)

(10) R_o = -(2 * h + 4) (i term)

(0) (j term)

(h + 4) (k term)

where Diophantion equation R_o is an identical solution of (8) for all values of the parameter h as may be seen by an actual substitution:

(11) R = R_o + t * G

= (21 * t - 2 * n - 4) (i term)

(0) (j term)

(h + 4 - 10 * t) (k term)

Put R into (4) and (5) and get:

$$(12) \quad 3 * h - 28 * t > -6$$

$$(13) \quad -15 * h + 159 * t > 34$$

whence

$$19 * t > 4$$

thus

$$t > 4/19 = 1$$

$$h > 22/3 > 7 = 8$$

then from (11)

R = (1 0 2) (integer minimum vector)

P = 44 + h = 44 + 8 = 52 (integer minimum)

also,

P = (10 14 21) * (1 0 2) = 52

We now solve this problem without any reference to a polyhedron of constraint.

A symbolic identity solution for the general objective function;

$$P = C * R$$

$$= (c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_N) * R$$

is given by:

$$R = (P / c_1 - t_2 - t_3 \ldots t_N) + (c_1 / c_2) * t_2 + (c_1 / c_3) * t_3 + \ldots (c_1 / c_N) * t_N$$

where t are just undefined parameters for the present. Later equations will define them. That the claimed solution is correct may be seen by an actual substitution.

For our equation $c_1 = 10$, $c_2 = 14$, $c_3 = 21$

$$R = (P/10 - t_2 - t_3) + (10/14) * t_2 + (10/21) * t_3$$

This equation enables us to put the objective function P into the constraint system giving us a lot of leverage power.

Put R into (4) and (5) and get:

(4)' $(P/5) - (4/7) * t_2 + (4/3) * t_3 > 14$

(5)' $(9P/10) - (2/7) * t_2 - (53/7) * t_3 > 10$

Eliminating t_3 from (4)' and (5)' we get

(6) $(19/7) * P - (212/49 - 8/21) * t_2 > 358/3$

Here we see that the smaller t_2 is the smaller is P. Setting

$$t_2 = 0$$

we get

$$P = 7 * 358/57 = 2506/57$$

and this is the same value as in the GAMMA solution:

$$P = (10 \quad 14 \quad 21) * (28 \quad 0 \quad 106)/57 = 2506/57$$

There was no mention of a polyhedron of constraint here. None was needed. We now evaluate the components of our symbolic vector R. Putting the values of P and t_2 into (4)' we get

$$t_3 = (14 - P/5) * (3/4)$$

but we need

$$X_3 = (10/21) * t_3 = (14 - P/5) * (10/28) = 106/57$$

$$X_1 = (P/10 - t_2 - t_3)$$

= 28/57

Thus

R = (28 0 106)/57 (same as GAMMA solution)

It should be pointed out that in minimization problems there may be a whole group of t that have to be set equal to zero. In this problem we only had to set one of them equal to zero. It is a criterion for a minimization. The criterion for a maximum is the setting of positive t terms equal to zero.

In GAMMA theory, the criterion for a minimum is that all products from the last index of C * GAMMA have to be positive, and for a maximum these products have to be negative.

The t criteria are just as valid or may be more so than those due to the GAMMA theory.

The symbolic vector brings a whole new approach to the whole field of linear programming.

It should have something to say when it is applied to the old Traveling Salesman problem. It seems made to order.