Brocard Generalization

By

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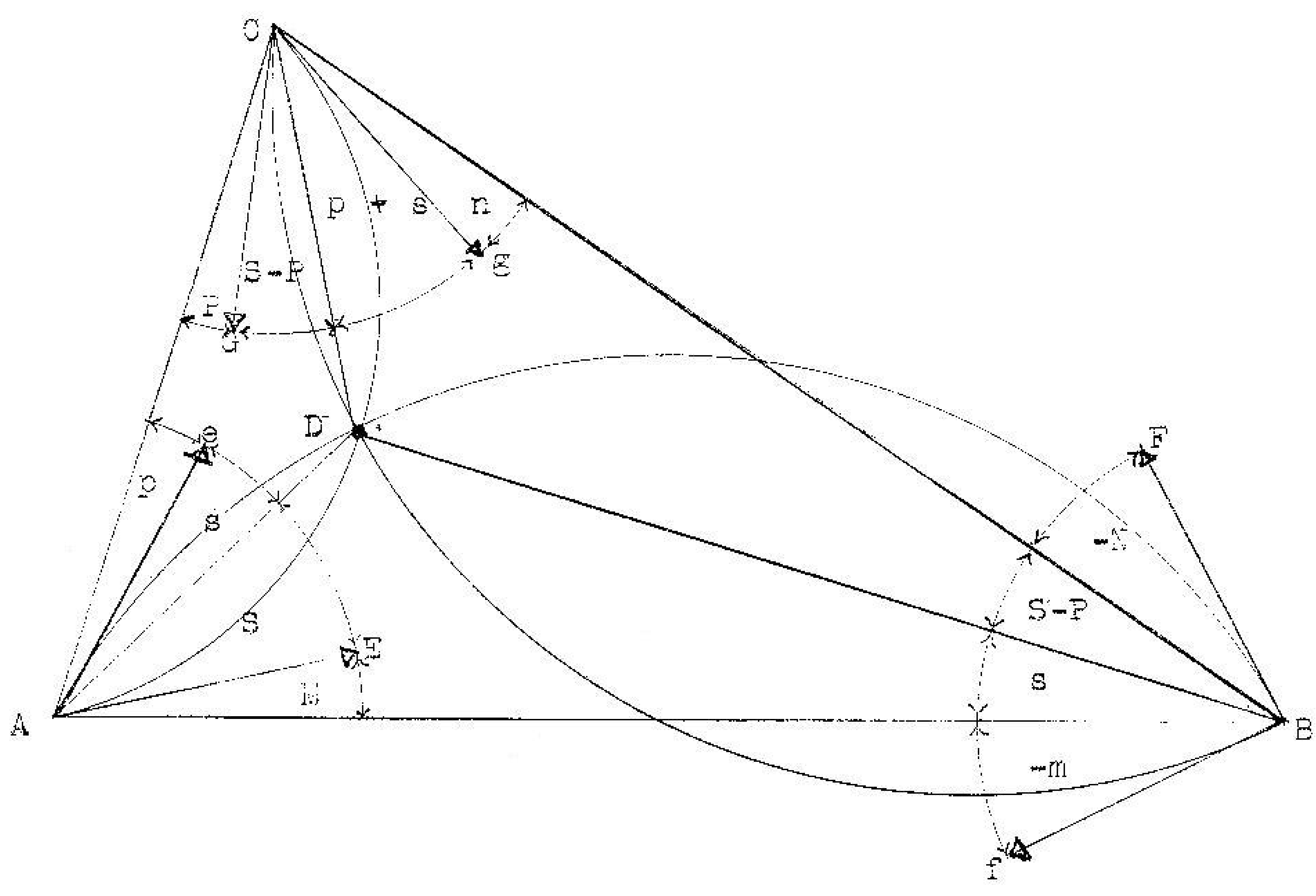
In this Generalization of the Brocard Theorems of the Direct and Indirect Groups of Ajoint Circles we shall show that the three cutting angles of the three circles ABD, BCD, and CAD, cutting the sides of the triangle ABC directly in the angles M, N, and P and indirectly in the angles p, n, and m are each given by the relations:

(1) $M + N + P = 0$

(2) $p + n + m = 0$

where those angles falling outside the triangle are counted as negative. See the drawing below.



Point D in triangle ABC is determined by the cutting angles M, N, and P. It may be made to take any position in the plane of the triangle by a proper choice of the size and sign of the cutting angles M, N, and P. For a convenient drawing we have considered it inside the triangle.

Construction

At A and B draw the tangents Ae and BF to the circle ABD meeting the sides AC and BC at angles p and -N respectively. At B and C draw tangents Bf and CG to the circle BCD meeting the sides AB and AC at angles -m and P respectively. At C and A draw tangents Cg and AE to the circle CAD meeting the sides BC and AB at angles n and M respectively. Going around the triangle ABC counter-clockwise the direct group of cutting angles is M, -N, and P. The indirect group of cutting angles is p, n, and -m. For ease in writing our equations we shall set angle DAE equal to S and angle DAe equal to s. We shall now prove relation (1).

Proof:

Consider arc ADC.

(3) $$\angle DCA = \angle DAE$$

since AE is tangent to the arc at A. Then

(4) $$\angle DCG + \angle P = \angle S \qquad \text{or}$$

(5) $$\angle DCG = \angle (S - P)$$

and is so marked in the figure. Now consider arc BDC.

(6) $$\angle DCG = \angle DBC$$

since CG is tangent to the arc at C. Then from (5) and (6)

(7) $$\angle DBC = \angle (S - P)$$

and is so marked in the figure. Lastly consider arc ADB.

(8) $$\angle DAE = \angle DBF$$

since BF is tangent to the arc at B. Then

(9) $$\angle S + \angle M = \angle (S - P) - \angle N \qquad \text{or}$$

(10) $$\angle (M + N + P) = 0$$

Equation (10) is the proof of our first relation. The proof of the indirect relation

$$(11) \qquad \angle(p + n + m) = 0$$

is identical with the direct except one starts with angle s instead of S and traverses the triangle ABC clockwise.

Critique

One obvious fact from (1) is that at least one of the angles M, N, and P must be postive. It is possible for two of them to be postive and the other negative as pictured in the drawing. As to which is postive it matters not. In any case (1) holds.

When

$$(12) \qquad \angle M = \angle N = \angle P = 0$$

we get the direct group of ajoint circles of the Brocard Theorem and point D coincides with the direct Brocard point of the triangle.

When

$$(13) \qquad \angle p = \angle n = \angle m = 0$$

we get the indirect group of ajoint circles of the Brocard Theorem and point D coincides with the indirect Brocard point of the triangle.

Every consequent deduction that comes from the Brocard Theorem has its counterpart in this Generalization and even more. We give an illustrative example. In college geometry it is proven that the direct and indirect Brocard points are isogonal congugate points of the triangle.

Let M, N, and P be the direct cutting angles corresponding to point D and AE, BF, and CG the corresponding tangents as shown in the above diagram. Now consider a second point D' not shown in the drawing, whose indirect cutting angles are p', n', and $-m'$, the corresponding tangents being Ae', Cg', and Bf'. Equation (2) holds for angles p', n', and $-m'$. Set angle DAe' equal to s'. Angles p', n', and m' are as yet wholy independent of the angles M, N, and P.

Perhaps it is not generally observed that there is a relation, in disguise, between the direct and indirect cutting angles of the circles for the isogonal congugate points of the Brocard Theorem namely that each direct cutting angle is equal to the corresponding indirect cutting angle each being zero. A natural Generalization for the Brocard isogonal congugate point theorem is to find the relation between the direct cutting angles M, N, and P and the indirect cutting angles p', n', and m' when D and D' are isogonal congugate points of the six tangents two at each vertex of triangle ABC since these coincide with the sides of the triangle ABC when D and D' coincide with the isogonal congugate Brocard points. We shall see.

Now

(14) $$\angle DCG = \angle (S - P)$$

(15) $$\angle D'Cg' = \angle (p' + s')$$

If D and D' are isogonals of the tangents CG and Cg' we must have

(16) $$\angle (p' + s') = \angle (S - P)$$

Again

(17) $$\angle D'Bf' = \angle DBA + \angle ABf' = \angle (s' - m')$$

and

(18) $$\angle DBF = \angle DBC + \angle CBF = \angle (S - P) - \angle N$$

If D and D' are isogonals of the tangents BF and Bf' we must have

(19) $$\angle (s' - m') = \angle (S - P) - \angle N$$

Adding (16) and (19) one gets

(20) $$2s' + \angle (p' - m') = \angle (2S - 2P) - \angle N$$

Adding $\angle (n' + 2m')$ to both sides of (20) one gets

(21) $$\angle 2s' + \angle (p' + n' + m') = \angle 2S + \angle (n' + 2m' - 2P - N)$$

If we set

(22) $$\angle (n' + 2m') = \angle (N + 2P)$$

and take account of (2) we get from (21)

(23) $\angle s' = \angle S$

From (16), (19) and (23) we see that D and D' are isogonal congugate points of the six tangent lines of the direct and indirect cutting circles. Put (23) into (16) and we get

(24) $\angle p' = - \angle P$

Put (23) into (19) and take account of (1) and we get

(25) $\angle m' = - \angle M$

Put (25) into (22) and take account of (1) and we get

(26) $\angle n' = - \angle N$

Now when the direct cutting angles M, N, and P are zero and the direct tangents collapse onto the sides of the given triangle and point D coincides with the direct Brocard point it is seen from (24), (25), and (26) that the indirect cutting angles p', n', and m' are zero and the indirect tangents collapse onto the sides of the given triangle ABC and point D' coincides with the indirect Brocard point thus showing that this particular corollary of the Brocard Theorem which we are illustrating is a very special case of a far greater Generalization. Other corollaries of the Brocard Theorem in a similar way may be shown to be particular cases of a more general viewpoint. All these follow from (1) which is the Generalization of the Brocard Theorem.

Note to the referee: This Generalization of the Brocard Theorem is one of four, Simson Line, Apollonian, and Feuerbachs which we have done from the Mutation standpoint and translated into convential geometry. An abstract of a paper on Mutation geometry given before the Ohio section of the Mathematical Assn. may be seen in the American Mathematical Monthly vol. 60 number 7 page 520 Aug. - Sept. 1953. As a closing observation I should like to call attention to the beautiful symmetry exhibited in (24), (25), and (26).