CHAPTER THREE

Circle

3 - 1. General Equation

A circle is the path of all points that are at an equal distance from a point called its center. We shall usuall designate the radius of a circle by a small letter as a, b, or c. If the center of the circle is at the origin we may designate its radius by r but not otherwise. We shall reserve the letter r for the radius vector to any point on a locus and so must not use it for the radius of the circle . See Fig. 3-1 for a visual sketch.

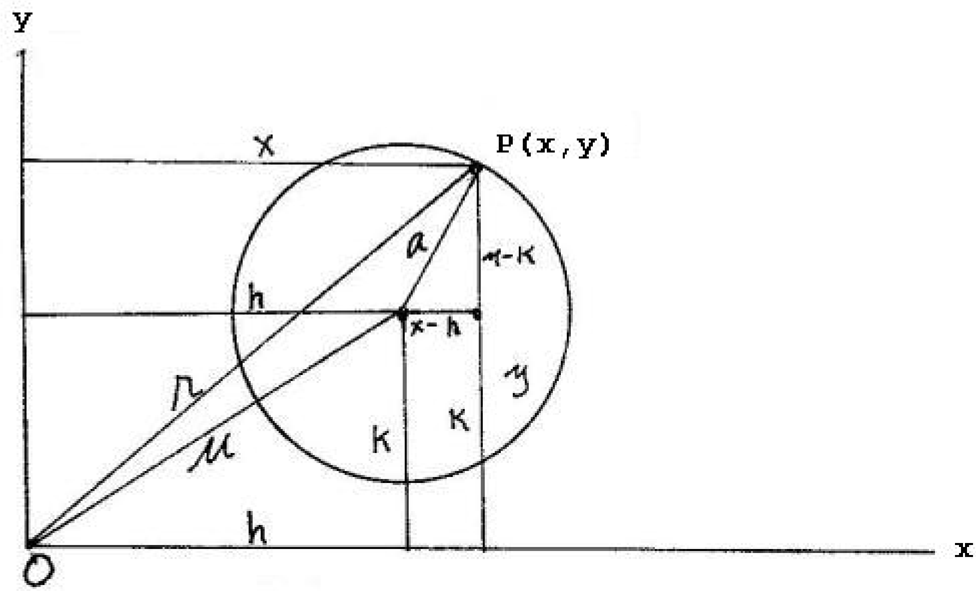


Fig. 3-1

The letter U will be used to represent the vector from the origin to the center of ALL conics and that includes the circle. Were it not for convention(and it is a strong thing) I would write my U in the following,way as usual :

$$U = U_1 i + U_2 j$$

or still better $$U = U_1 i_1 + U_2 i_2$$

and $$r = x_1 i_1 + x_2 i_2$$

but to sort of compromise with convention we shall write them

$$U = h\,i + k\,j$$

$$r = x\,i + y\,j$$

To find the equation of the circle with center at (h, k) and radius a we have from **Fig. 3-1** the relation:

$$(1) \qquad (r - U)^2 = a^2$$

If we put in the expressions for r and U we get:

$$(2) \qquad ((x - h)i + (y - k)j)^2 = a^2$$

Performing the indicated operation and remembering that the cross products are zero since i and j are perpendicular to each other we get for the cartesian equation of a circle:

$$(3) \qquad (x - h)^2 + (y - k)^2 = a^2$$

This equation confirms the right triangle in the figure with sides $x - h$ and $y - k$ and hypotenuse a. One could have used this fact to derive the equation of the circle. A geometric minded person might even prefer it. All power to them.

When h and k are both zero equation (3) reduces to the more simple form:

$$(4) \qquad x^2 + y^2 = a^2$$

In this case the origin is at the center of the circle. In that case equation (1) becomes:

$$(5) \qquad r^2 = a^2$$

or what is the same thing

$$(6) \qquad r_0 = a$$

In analogy with (6), (1) may be written in the very useful form:

$$(7) \qquad (r - U)_0 = a$$

All circles are not written like (3) but they may be reduced to that form. For instance consider the equation

$$x^2 + y^2 - 4x - 6y = 3$$

By adding 13 to both sides of the equation we can put it in the form

$$x^2 - 4x + 4 + y^2 - 6y + 9 = 16$$

which may be written in the for

$$(x - 2)^2 + (y - 3)^2 = 4^2$$

which is in the form of (3). It is a circle whose center is at (2, 3) and with radius 4.

One should observe that if the squaring indicated in (3) is performed no x y term can ever appear. It should also be observed that the coefficients of the squared terms are always equal and may be made unity.

Exercises

Write the equations of the following circles:

1. With center at (2, 1) a nd radius 3
2. With center at (3,-2) and radius 6
3. With center at (-4,-5) and radius 2
4. With center at (7, 8) and passing thru (3, 5)
5. With center a t (2, 4) and tangent to $4x + 3y = 10$

Find the center and radius of the following circles

6. $x^2 + y^2 - 2x - 4y = 11$
7. $x^2 + y^2 - 4x - 6y = 12$
8. $x^2 + y^2 - 2x - 2y = 7$

From high school geometry one learned that three non-coincident points always determined a circle, even if they were in a straight line, the circle in that case being identifyed with the line and having an infinite radius. That is too large a radius. What we want to emphasize is not the large radius but that three conditions determine a definite circle. Three points determine a circle, two points and a given tangent determine a circle, three tangents determine a circle. In general three conditions determine a circle. In equation (2) we have three unknowns h, k, and a and it takes three equations to solve for the three unknowns. That means we skall have to have three different sets of values of x and y in which case we may form three different equations from equation (2) by putting these different values of x and y into it. We shall soon do a number of illustrative examples covering this discussion.

The equation of a circle may also be written in the form:

$$(8) \qquad x^2 + y^2 = Dx + Ey + F$$

for one could transpose all terms to the left and complete squares and get it in the form of equation (3).

Example 1

Find the equation of a circle thru the three pointd: (2, - 2), (- 2, 0), and (- 1, - 1). If we substitute these coordinates for the x and y in equation (8) we get the following three equations :

$$2D - 2E + F = 8$$

$$-2D + 0E + F = 4$$

$$-1D - 1E + F = 2$$

Solving these for D, E, and F we get: $D = 4$, $E = 6$, $F = 12$. Putting these values into (8) we get for the equation of our circle:

$$x^2 + y^2 - 4x - 6y = 12$$

This may be put into the form of (3):

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

This last solution is one of the ways of conventional geometry to solve the problem. It is not a bad way to get a solution. The worst part seems to be in having to solve the system of three equations.

We now look at it from a slightly different viewpoint. Let b, c, and d be the vectors from the origin to the three given points on the circumference of the circle. We may now write the following three equations:

$$(b - U)^2 = a^2$$

$$(c - U)^2 = a^2$$

$$(d - U)^2 = a^2$$

If we subtract the second and third from the first we get the two following proto-type equations:

$$2(b - c) \cdot U = b^2 - c^2$$

$$2(b - d) \cdot U = b^2 - d^2$$

According to (43) chapter 1, U is given by:

$$(9) \qquad U = \frac{(b^2 - c^2)(b - d)^{\vee} - (b^2 - d^2)(b - c)^{\vee}}{2(b - c) \cdot (b - d)^{\vee}}$$

Putting in the values of b, c, and d we get for U the value

$$U = 2i + 3j$$

Put this value of U back into any of the first three equations and we get

$$a^2 = 5^2$$

Equation (1) now becomes

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

3 - 2 A Pattern Of Thought

Equation (9) is an interesting expression and superficially seems somewhat involved but it is not true. Write down the three points in a column and the answer for U below it as:

$$\begin{array}{rr} 2 & -2 \\ -2 & 0 \\ -1 & -1 \end{array}$$

$$U = \frac{4(1 + 3) - 6(2 + 4)}{-4} = (2, 3).$$

The 4 before the first parenthesis is the difference between the square of the first and second rows 8 - 4. The 1 in the first parenthesis is the difference between the right coordinate in the third row and the right coordinate in the first row -1 -(-2). The 3 in the first parenthesis is the difference between the left coordinate in the first row and the left coordinate in the third row 2 - (- 1). The 6 before the second parenthesis is the difference between the squares of the first and third rows 8 - 2 . The first number 2 in the second parenthesis is the difference between the right coordinate in the second row and the right coordinate in the first row 0 - (- 2) . The 4 in the second parenthesis is the difference between the left coordinate in the first row and the left coordinate in the second row 2 - (- 2). To get the - 4 in the denominator we look at the two parentheses with their numbers and signs ONLY, other numbers and their signs having no effect. We take the product of the two outside numbers 1 and 4 and the product of the inside numbers 3 and 2 and double the difference between these two products.

To get the two answers we take

$$\frac{4(1) - 6(2)}{-4} = 2$$

$$\frac{4(3) - 6(4)}{-4} = 3$$

To obtain the radius a of the circle we have three choices. Inspection shows that we ought to subtract the U row from the b row or first row for the 2,s cancel and we have 5 left which is the radius of the circle. and the equation of the circke may be easily written

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

We have tediously gone thru this pattern of thinking firstly, because the student has had little practice in reading mutation equations and secondly one should learn to think of what is happening instead of mechanically following a formula. One should not look at formula (9). One ought to be able to write the equation of the circle by looking at the,column of the coordinates. Let us take a new example. Do not look at (9). Find the equation of a circle thru the three points whose coordinates are written in a column below:

$$\begin{array}{cc} -1 & 5 \\ -4 & 2 \\ 2 & 2 \end{array}$$

$$U = \frac{6(-3-3) - 18(-3+3)}{-36} = (-1, 2)$$

$$(x+1)^2 + (y-2)^2 = 3^2$$

We do a second example, and one should be able to simply write the equation from the column of the coordinates. Do not look at equation (9). It is not INDRA. Find the equation of a circle passing thru the three points whose coordinates are written in a column below.

$$\begin{array}{cc} -2 & 5 \\ -3 & 4 \\ -1 & 4 \end{array}$$

$$U = \frac{4(-1-1) - 12(-1+1)}{-4} = (-2, 4)$$

$$(x+2)^2 + (y-4)^2 = 1^2$$

Find the circle thru the three points (1, 5), (2, - 2), and (0, 2).

$$U = \frac{18(-3+1) - 22(-7-1)}{20} = (5,2)$$

$$(x-5)^2 + (y-2)^2 = 5^2$$

Exercises

Find the equations of the circles passing thru each of the five sets of triple points:

1	2	3	4	5
- 6, - 1	1, 6	3, - 6	6, 0	3 - 2
- 3, 2	5, 2	3, 4	2, 4	- 2 3
0, - 1	- 3, 2	- 2, - 1	- 2, 0	- 1 0

3 - 3 Tangent Circles

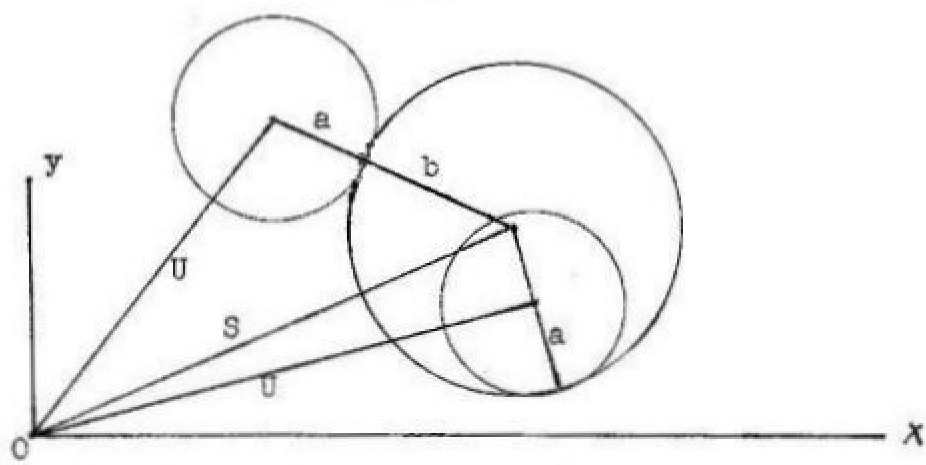


Fig. 3 - 2

See Fig. 3 - 2 . If the distance between the cemters of two circles is equal to the sum or difference of their radii the circles are tangent. This is self evident and needs no proof. To conjure up a proof would be a gross waste of time. See the drawing for the " Proof ". This is an Oliver Heaviside proof. " it works " .

The condition that the two corcles be tangent may be written

$$(1) \qquad (U - S)^2 = (a \pm b)^2$$

where U and S are the vectors from the origin to the centers of thw two circles and a ahd b are their radii. If the equations of the two circles are given these determine the quantities U, S, a, and b. We may then put these values into the equation and see if they satisfy. We shall look at a couple of illustrative examples to get some sense and feeling about the matter.

Test whether the following two circles are tangent.

$$(x - 1)^2 + (y - 2)^2 = 4^2$$
$$(x + 3)^2 + \phi y + 1)^2 = 1^2$$

$$U = i + 2j$$
$$S = -3i - 1j$$
$$U - S = (4i + 3j)$$
$$a + b = 4 + 1 = 5$$
$$(U - S)^2 = 25 = (a + b)^2$$

Thus the circles are tangent.

Example 2

Test the two circles below for tangency.

$$(x - 2)^2 + (y - 3)^2 = 5^2$$
$$(x + 3)^2 + (y + 1)^2 = 3^2$$
$$U = 2i + 3j$$
$$S = -3i - 1j$$
$$(U - S)^2 = (5i + 4j)^2 = 41$$
$$(a + b)^2 = 64$$

$$(a - b)^2 = 4$$

Neither 4 nor 64 is equal to 41 whiah shows that equation (1) is not satisfied and thus the two circles are not tangent.

Exercises

Test the the following three sets of circles for tangency

1
$$(x - 2)^2 + (y - 8)^2 = 15^2$$
$$(x + 3)^2 + (y + 4)^2 = 2^2$$

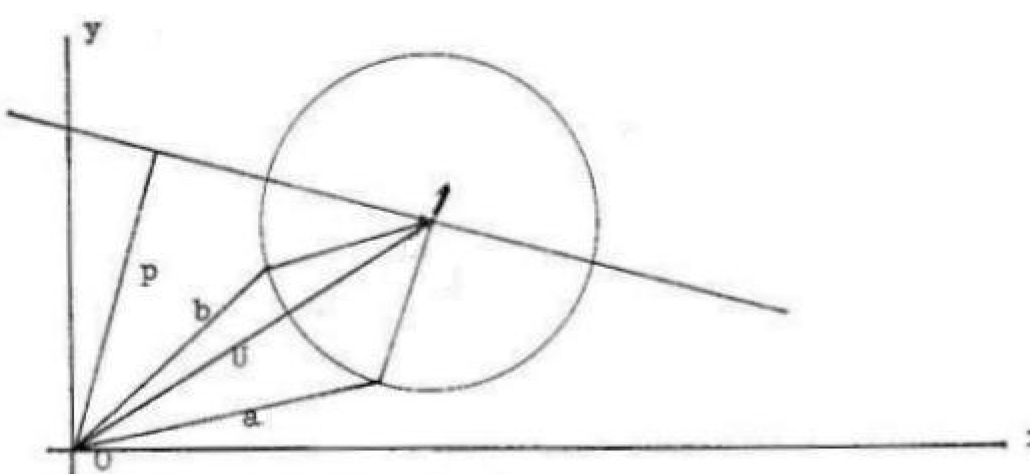
2
$$(x - 1)^2 + (y + 5)^2 = 8^2$$
$$(x + 1)^2 + (y + 4)^2 = 3^2$$

3
$$(x - 4)^2 + (y - 2)^2 = 7^2$$
$$(x + 2)^2 + (y + 6)^2 = 3^2$$

3 - 4 Equation of a Circle Thru Two Points with Center on Given Line

Let the given points be designated by the vectors a and b. Let the given line be represented by the normal equation :

$$(1) \qquad p' \cdot r = p_o$$



Fig, 3 - 3

Let U be the vector to the center of the circle which is on the given line. See Fig. 3 - 3 for details. If one produces the line U - b twice itself one gets the diameter of the circle thru the end of vector b. If this diameter is projected on the vector a - b we get the magnitude of a - b. Thus we can write the equation*

$$(2) \qquad 2 (a - b) \cdot (U - b) = (a - b)^2$$

Since U satisfies equation (1) we can write

$$(3) \qquad p' \cdot U = p_0$$

Subtracting $p' \cdot b$ from both sides of (3) we get

$$(4) \qquad p' \cdot (U - b) = p_0 - p' \cdot b$$

Solving (2) and (4) for the vector (U - b) according to equation (11) 1 - 5 Of chapter I we get; after transposing the b:

$$(5) \qquad U = b - \frac{(p_0 - p' \cdot b) (a - b)^{\breve{}} + ((a - b)^2/2) \breve{p}'}{(a - b) \cdot \breve{p}'}$$

Having found the value of U we may now write the equation of our circle. It is :

$$(6) \qquad (x - h)^2 + (y - k)^2 = (U - b)^2$$

We do a couple of numerical examples just to get the feel.

Example 1.

Find the equation of a circle thru the two points (3, 2), and (4, 3) whose center is on the line whose equation is x + 2 y = 11.

$$a = 3 i + 2 j \qquad b = 4 i + 3 j$$

$$a - b = - (i + j) \qquad (a - b)^2 = 2$$

$$(a - b)^{\breve{}} = i - j \qquad p' = (i + 2 j)/\sqrt{5}$$

$$p' \cdot b = 10/\sqrt{5} \qquad (a - b) \cdot \breve{p}' = 1/\sqrt{5}$$

$$p_0 = 11/\sqrt{5} \qquad \breve{p}' = (- 2 i + j)/\sqrt{5}$$

Putting this table of values into equation (5) we get

$$U = i + 5j, \quad U - b = -3i + 2j$$

$$h = 1, \quad k = 5, \quad (U - b)^2 = 13$$

$$(x - 1)^2 + (y - 5)^2 = 13 \qquad (7)$$

This equatfon (7) satisfies all the requirements and is the required equation.

Example 2.

Find the equation of a circle passing thru the two points (2, - 2), (- 1, - 1) whose center is on the line $-x + y = 1$.

$$a = 2i - 2j \qquad p' = (-i + j)/\sqrt{2}$$

$$b = -1i - 1j \qquad \overset{\smile}{p} = -(i + j)/\sqrt{2}$$

$$a - b = 3i - 1j \qquad p' \cdot b = 0$$

$$(a - b)^{\smile} = 1i + 3j \qquad p_0 = 1/\sqrt{2}$$

$$(a - b)^2 = 10 \qquad (a - b) \cdot \overset{\smile}{p} = -2/\sqrt{2}$$

Put this table of values into (5) and we get the following:

$$U = 2i + 3j \qquad (U - b)^2 = 25$$

Thus our required circle becomes:

$$(x - 2)^2 + (y - 3)^2 = 25. \qquad (8)$$

Exercises

1 Find the equation of a circle thru the two points (0, 2), and (4, 4) with its center on the line $2x + y = 3$

2 Find the equation of a circle thru (5, 2), and (0, + 3), with its center on the line $x + y = 2$.

3-5 Equation of a Circle Thru a Given Point Tangent to a Given Line and with it's Center on a Given Line.

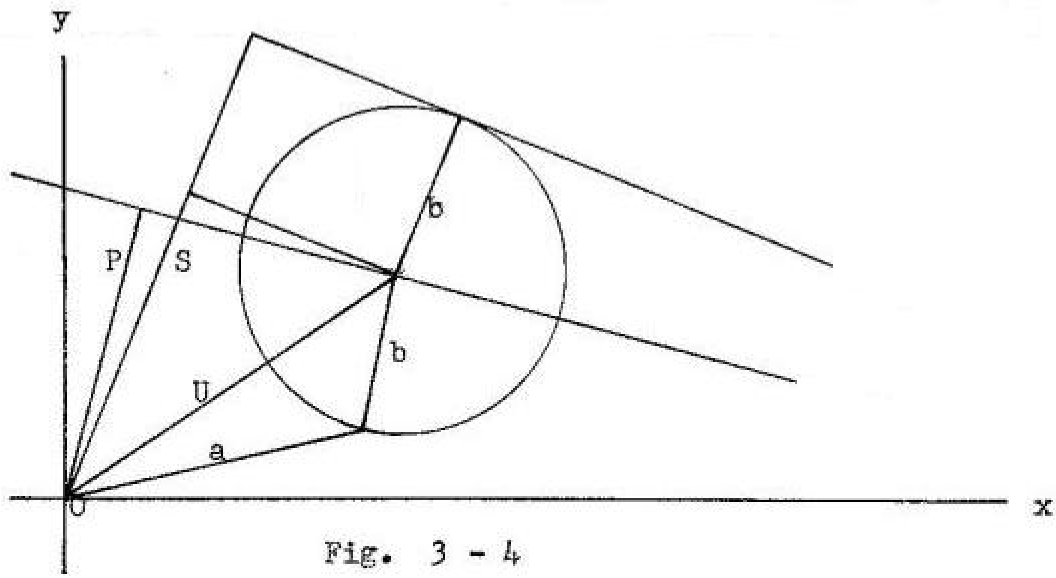


Fig. 3 - 4

Let a be the vector to the given point. Let the equations of the tangent and line thru its center be given respectively by:

$$(1) \qquad S' \cdot r = S_0$$

$$(2) \qquad P' \cdot r = P_0$$

$$(3) \qquad (U - a)_0 = b$$

where b is the radius of the required circle and U is the vector from the origin to its center. From the sketch we may write the following equations:

$$S' \cdot U + (U - a)_0 = S_0$$

Subtracting $S' \cdot a$ from both sides of this equation we get.

$$(4) \qquad S' \cdot (U - a) + (U - a)_0 = S_0 - S' \cdot a = M$$

$$P' \cdot U = P_0$$

Subtracting $P' \cdot a$ from this last equation we get:

$$(5) \qquad P' \cdot (U - a) = P_0 - a \cdot P' = N$$

Dividing (4) by (5) thus cancelling the magnitude of (U - a) and clearing of fractions we get:

$$(6) \qquad d \cdot (U - a)' = 1$$

$$d = n P' - S', \qquad n = M/N$$

From (6), according to (2) 1 - 2 chapter I, we get:

$$(7) \qquad (U - a)' = (d \pm \breve{d} \sqrt{d^2 - 1}) \, d^{-2}.$$

Multiplying this last equation by P' we get:

$$(8) \qquad P' \cdot (U - a)' = P' \cdot (d \pm \breve{d} (d^2 - 1)^{1/2}) d^{-2} = L$$

Divide (5) by (8) and we get

$$(9) \qquad (U - a)_0 = N/L$$

Multiplying (7) by (9) and transposing the a and we get:

$$(10) \qquad U = a + (N/L) (d \pm \breve{d} \sqrt{d^2 - 1}) \, d^{-2}$$

We can now write our equation:

$$(11) \qquad (x - h)^2 + (y - k)^2 = (U - a)^2 = (N/L)^2$$

W e shall do a numerical for this problem to see in detail how the New Styling of Mutation Geometry does its problems.

Example 1.

Find the equation of a circle thru (- 1, - 1), tangent to the line whose equation is

$$x + 2y = 8 + 5\sqrt{5}$$

and whose center is on the line whose equation is

$$x + y = 5.$$

In this case we can write the following relations:

$$a = -(i + j)$$

$$P' = (i + j)/\sqrt{2}$$

$$S' = (i + 2j)/\sqrt{5}$$

$$S_0 = (8 + 5\sqrt{5})/\sqrt{5}$$

$$S' \cdot a = -3/\sqrt{5}$$

$$S_0 - S' \cdot a = (11 + 5\sqrt{5})/\sqrt{5} = M$$

$$P_0 = 5/\sqrt{2}$$

$$P' \cdot a = -2/\sqrt{2}$$

$$P_0 - P' \cdot a = 7/\sqrt{2} = N$$

$$n = M/N = (11 + 5\sqrt{5})\sqrt{2}/7\sqrt{5}$$

$$n P' = ((11 + 5\sqrt{5})/7\sqrt{5})(i + j)$$

$$d = nP' - S' = ((4 + 5\sqrt{5})i + (-3 + 5\sqrt{5})j)/7\sqrt{5}$$

$$\breve{d} = ((3 - 5\sqrt{5})i + (4 + 5\sqrt{5})j)/7\sqrt{5}$$

$$d^2 = (275 + 10\sqrt{5})/245$$

$$d^2 - 1 = (30 + 10\sqrt{5})/245$$

$$(d^2 - 1)^{1/2} = \pm(5 + \sqrt{5})/7\sqrt{5}$$

$$\breve{d}(d^2 - 1)^{1/2} = (-(10 + 22\sqrt{5})i + (45 + 29\sqrt{5})j)/245.$$

$$(U - a)' = (d \pm \breve{d} (d^2 - 1)^{1/2}) d^{-2}$$
$$= (\frac{(165 + 6\sqrt{5}) i + (220 + 8\sqrt{5}) j}{(275 + 10\sqrt{5})})$$

$$(U - a)' \cdot P' = \frac{385 + 14\sqrt{5}}{\sqrt{2} (275 + 10\sqrt{5})} = L$$

$$(U - a)_0 = N/L = \frac{275 + 10\sqrt{5}}{55 + 2\sqrt{5}}$$

$$(U - a) = \frac{(165 + 6\sqrt{5}) i + (220 + 8\sqrt{5}) j}{55 + 2\sqrt{5}}$$

$$U = 2 i + 3 j , \qquad b = S_0 - S' \cdot U = 5$$

We now write one of our equations:

$$(x - 2)^2 + (y - 3)^2 = 5^2.$$

For this equation we took the plus sign in the proto-type solution. When we use the negative sign we get the other equation:

$$h = (28 + 10\sqrt{5})/9, \qquad k = (17 - 10\sqrt{5})/9 ,$$
$$b = (55 + 2\sqrt{5})/9 .$$

This second equation is slightly ugly. It would be too much to expect everything to be to our liking. It is as good as the first one.

3 - 6 A Generalization

When one talks about a circle being tangent to a given line it is implied thst the circle cuts the given line either at zero or 180 degrees. We wish now to solve the same problem as above except that the required circle is to cut a given line at a specified angle. See Fig. 3- 5 .

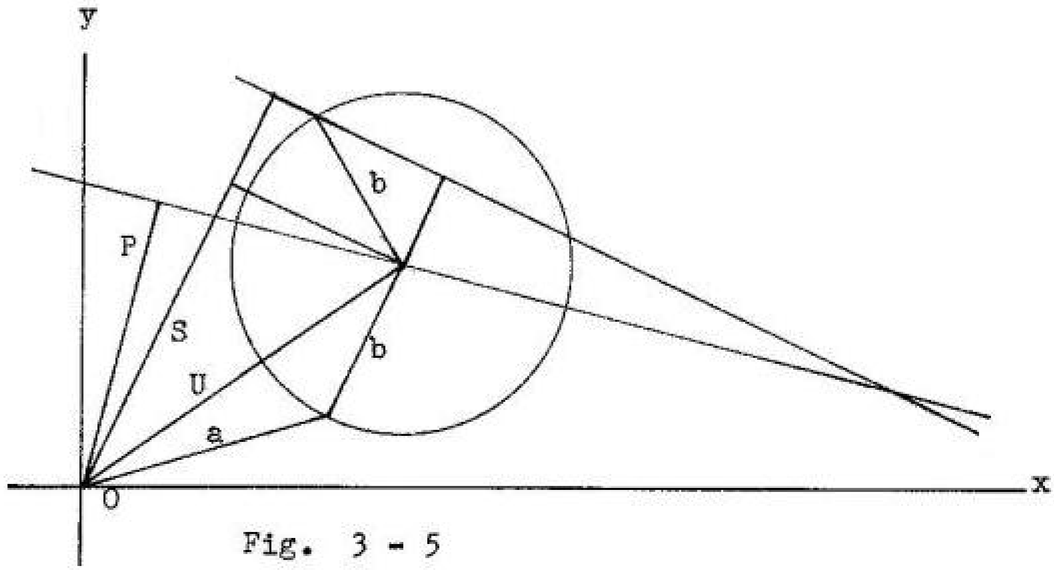


Fig. 3 - 5

Find the equation of a circle passing thru a given point, cutting a given line at a given angle and having its center on a given line.

Let a be the vector to the given point. Let P and S be the vector perpendiculars to the line thru the center of the circle and the cutting line respectively. Let b be the radius of the required circle. Let U be the vector to the center of the required circle.

From high school geometry one knows that the angle between two tangents to two circles at their point of intersection is equal to the angle between their radii drawn to the point of intersection. Let e be the cos of the cutting angle then the perpendiculatr distance from the center of the circle to the cutting chord is e b. From the figure then we may write the following equations:

$$(1) \qquad (U - a)_0 = b$$

$$(2) \qquad S' \cdot U + e\,(U - a)_0 = S_0$$

Subtracting $S' \cdot a$ from both sides of (2) we get:

$$(3) \qquad S' \cdot (U - a) + e\,(U - a)_0 = S_0 - S' \cdot a = M$$

$$(4) \qquad P' \cdot U = P_0$$

Subtracting $P' \cdot a$ from both sides if this equation we get:

$$(5) \qquad P' \cdot (U - a) = P_0 - P' \cdot a = N$$

Dividing (3) by (5) and rearranging we get:

$$(6) \qquad d \cdot (U - a)' = e$$

where d is the same as in the last developement. The only difference in this equation (6) and the equation (6) Of the last developement is that the 1 there has been replaced by the e in this equation. Each step in both problems from here on to the end is the same, the content of the prototype solutions, of course, being slightly different. The solution for the prototype above is

$$(7) \qquad (U - a)' = (e\,d \pm \breve{d}\sqrt{d^2 - e^2})\,d^{-2}.$$

Multiplying both sides of (7) by P' we get:

$$(8) \qquad P' \cdot (U - a)' = P' \cdot (e\,d \pm \breve{d}\sqrt{d^2 - e^2})\,d^{-2} = L.$$

Divide (5) by (8) and we get :

$$(9) \qquad (U - a)_0 = N/L$$

Multiply (7) by (9), getting $(U - a)$, then transposing the a and we get a desired result:

$$(10) \qquad U = a + (N/L)(e\,d \pm \breve{d}\sqrt{d^2 - e^2})\,d^{-2}$$

$$(11 \qquad (x - h)^2 + (y - k)^2 = (N/L)^2.$$

$$(12) \qquad (x - H)^2 + (y - K)^2 = (\quad)^2$$

The last parenthesis is to be filled in with corresponding numbers when the sign in the proto-type solution is changed.

Example 1 .

Find the equation of a circle passing thru (- 1, - 1), cutting the line whose equation is

$$x + 2y = 8 + 2.5\sqrt{5}$$

at 60 degrees and having its center on the line whose equation is

$$x + y = 5$$

We may them write the following relations:

$$a = -(i + j)$$
$$S' = (i + 2j)/\sqrt{5}$$
$$S_0 = (8 + 2.5\sqrt{5})/\sqrt{5}$$
$$S' \cdot a = -3/\sqrt{5}$$
$$S_0 - S' \cdot a = (11 + 2.5\sqrt{5})/\sqrt{5} = M$$
$$P_0 = 5/\sqrt{2}$$
$$P' \cdot a = -2/\sqrt{2}$$
$$P' = (i +, j)/\sqrt{2}$$
$$P_0 - P' \cdot a = 7/\sqrt{2} = N$$
$$n = M/N = \sqrt{2}(11 + 2.5\sqrt{5})/7\sqrt{5}$$
$$d = nP' - S' = \frac{(4 + 2.5\sqrt{5})i + (-3 + 2.5\sqrt{5})j}{7\sqrt{5}}$$
$$\breve{d} = \frac{(+3 - 2.5\sqrt{5})i + (4 + 2.5\sqrt{5})j}{7\sqrt{5}}$$
$$d^2 = \frac{87.5 + 5\sqrt{5}}{245}$$
$$d^2 - e^2 = \frac{26.25 + 5\sqrt{5}}{245}$$

$$\sqrt{d^2 - e^2} = (5 + 0.5\sqrt{5})/7\sqrt{5}$$

$$e\,d = \frac{(4 + 2.5\sqrt{5})i + (-3 + 2.5\sqrt{5})j}{14\sqrt{5}}$$

$$\breve{d}\sqrt{d^2 - e^2} = \frac{(17.5 - 22\sqrt{5})i + (52.5 + 29\sqrt{5})j}{490}$$

$$e\,d + \breve{d}\sqrt{d^2 - e^2} = \frac{(105 + 6\sqrt{5})i + (140 + 8\sqrt{5})j}{490}$$

$$(U - a)' = \frac{(e\,d + \breve{d}\sqrt{d^2 - e^2})}{d^2} = \frac{(105 + 6\sqrt{5})i + (140 + 8\sqrt{5})j}{175 + 10\sqrt{5}}$$

$$P' \cdot (U - a)' = \frac{245 + 14\sqrt{5}}{\sqrt{2}\,(175 + 10\sqrt{5})} = L$$

$$(U - a)_0 = \frac{175 + 10\sqrt{5}}{35 + 2\sqrt{5}} = 5$$

$$(U - a) = \frac{(105 + 6\sqrt{5})i + (140 + 8\sqrt{5})j}{(35 + 2\sqrt{5})}$$

$$U = \frac{(70 + 4\sqrt{5})i + (105 + 6\sqrt{5})j}{(35 + 2\sqrt{5})} = 2i + 3j$$

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

is the first of the two answers. The other answer may be obtained by using the negative sign in the proto-type solution.

3 - 7 A Second Generalization

A line passing thru the center of a circle cuts that circle at 90 degrees. We wish now to find the equation of a circle passing thru a given point and cutting two given lines at given angles.

Find the equation of a circle passing thru a given point and cutting two given lines at given angles. See Fig. 3 - 6 .

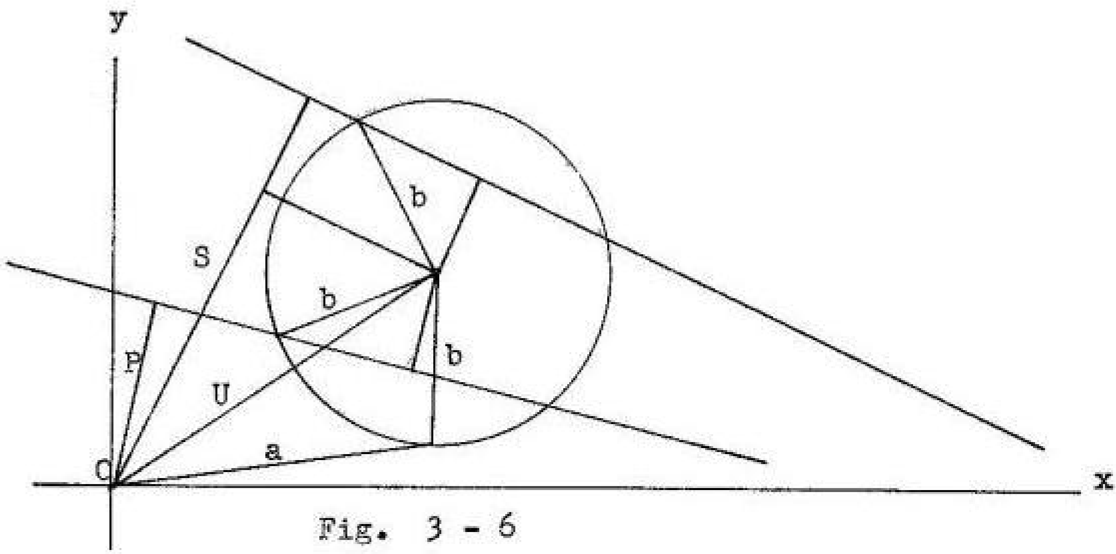


Fig. 3 - 6

Let a be the vector to the given point. Let P and S be the perpendiculars to the two given cutting lines. Let e and f be the cosines of their cutting angles respectively. Let U be the vector to the center of the required circle. We may then write the following equations:

$$(1) \quad (U - a)_0 = b$$

$$(2) \quad P' \cdot U + e(U - a)_0 = P_0$$

The last equation may be written:

$$(3) \quad P' \cdot (U - a) + e(U - a)_0 = P_0 - P' \cdot a = M$$

and in identically we write for the other cutting line:

$$(4) \quad S' \cdot (U - a) + f(U - a)_0 = S_0 - S' \cdot a = N$$

Dividing (3) by (4) and rearranging we get:

$$(5) \quad c \cdot (U - a)' = d$$

$$(6) \quad c = P' - n S'$$

(7) $d = e - nf$

(8) $n = M/N$

Solving (5) we get:

(9) $(U - a)' = (dc \pm \breve{c}\sqrt{c^2 - d^2})\,c^{-2}$

Multiplying (9) by P' and adding e we get:

(10) $P' \cdot (U - a)' + e = P' \cdot (dc \pm \breve{c}\sqrt{c^2 - d^2})\,c^{-2} + e = L$

Divide (3) by (10) and we get:

(11) $(U - a)_0 = M/L$

Multiply (9) by (11) giving us (U - a) then transposing the a and we obtain:

(12) $U = a + (M/L)(dc \pm \breve{c}\sqrt{c^2 - d^2})\,c^{-2}$

We can now write the two required equations:

(13) $(x - H)^2 + (y - K)^2 = (M/L)^2$

(14) $(x - h)^2 + (y - k)^2 = (m/l)^2$

It should be noticed that the theory for the generalizations is everywhere parallel to that for the particular cases. It is a remarkable thing that one does not have to shift mental gears in going from particular cases to generalizations. It is a salient characteristic of Mutation Geometry.

These generalizations here are only a foretaste of the complete generalizations of all the essential theorems of college geometry and be it added without any " shift of gear ". After that the projective field, the most general geometry of civilized man.

We shall now generalize the problem of section 3 - 4 in which it was required to find a circle passing thru two given points with its center on a given line.

3 - 8 Equation of a Circle thru two Points and Cutting Gimen Line at a Given Angle.

Find the equation of a circle passing thru two given points and cutting a given line at a given angle. See Fig, 3 - 7

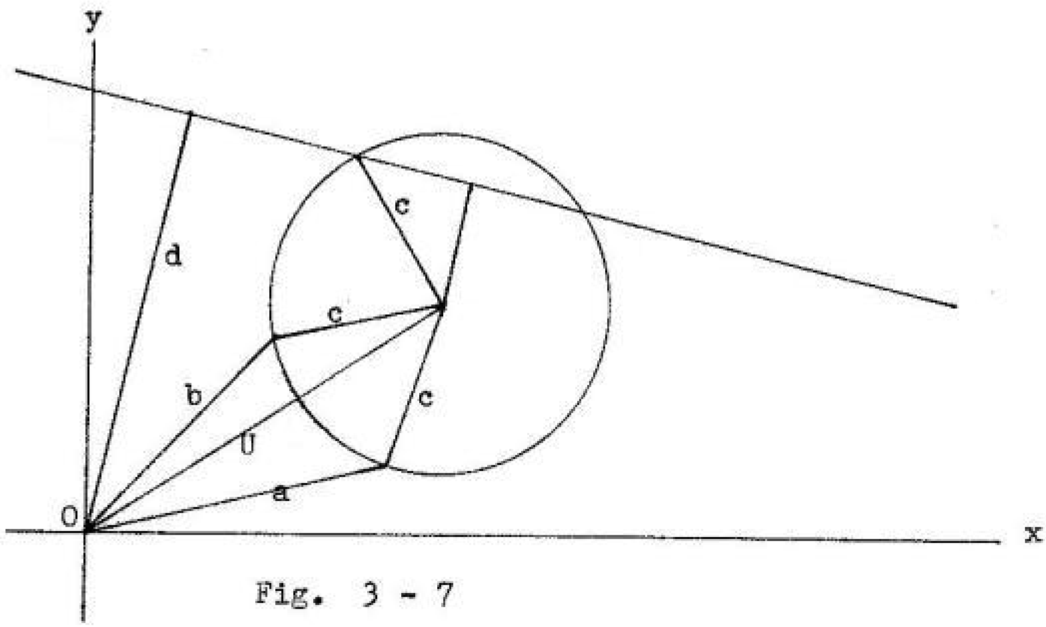


Fig. 3 - 7

Let a and b be the vectors to the given points, c the radius of the required circle and d the vector perpendicular distance from the origin to the given line. Let e be the cos of the cutting angle. Let U be the vector from the origin to the center of the required circle. From the figure we may write the following equations:

$$(1) \qquad (U - a)^2 = (U - b)^2 = c^2$$

$$(2) \qquad d' \cdot U + e(U - a)_0 = d_0$$

Subtracting $d' \cdot a$ from both sides of (2) we get:

$$(3) \qquad d' \cdot (U - a) + e(U - a)_0 = d_0 - d' \cdot a = M$$

From (1) we get, after squaring and rearranging.

$$(4) \qquad 2(b-a)\cdot(U-a) = (b-a)^2 = N^2$$

Dividing (3) by (4) and simplifying we obtain:

$$(5) \qquad n\cdot(U-a)' = e$$

$$(6) \qquad n = fg - d'$$

$$(7) \qquad f = M/N^2, \qquad g = 2(b-a)$$

Solving (5) we get:

$$(8) \qquad (U-a)' = (en \pm \breve{n}\sqrt{n^2-e^2})\,n^{-2}$$

Multiply (8) by g and we obtain:

$$(9) \qquad g\cdot(U-a)' = g\cdot(en \pm \breve{n}\sqrt{n^2-e^2})\,n^{-2} = L$$

Divide (4) by (9) and we obtain:

$$(10) \qquad (U-a)_0 = N^2/L$$

Multiply (8) by (10), getting (U - a) then transposing the a and we obtain the vector to the Center of the circle:

$$(11) \qquad U = a + (N^2/L)(en \pm \breve{n}\sqrt{n^2-e^2})\,n^{-2}.$$

From this last equation we may write the answers:

$$(12) \qquad (x-H)^2 + (y-K)^2 = (N^2/L)^2$$

$$(13) \qquad (x-h)^2 + (y-k)^2 = (\qquad)^2$$

The last parenthesis is to be filled in with corresponding numbers when the sign in the proto-type solution is changed.

3 - 9 Tangents to a Given Circle

Tangents may be drawn from a given point to a given circle provided the given point is not inside the given circle. If the point is outside the circle two tangents may be drawn from it to the given circle. If the point is on the circumference of the circle then the two tangents coincide and become one in local. They both may point in the same direction or one may point in one direction and the other in the opposite direction.

Since we are more interested in generalizations in the New Science of Mutation Geometry it would be better to talk about lines thru a given point cutting a given circle at a given angle or of lines cutting a circle at a given angle that have a specified direction, the tangents appearing as special cases of the more general viewpoint. This we shall do. See Fig. 3 - 8 .

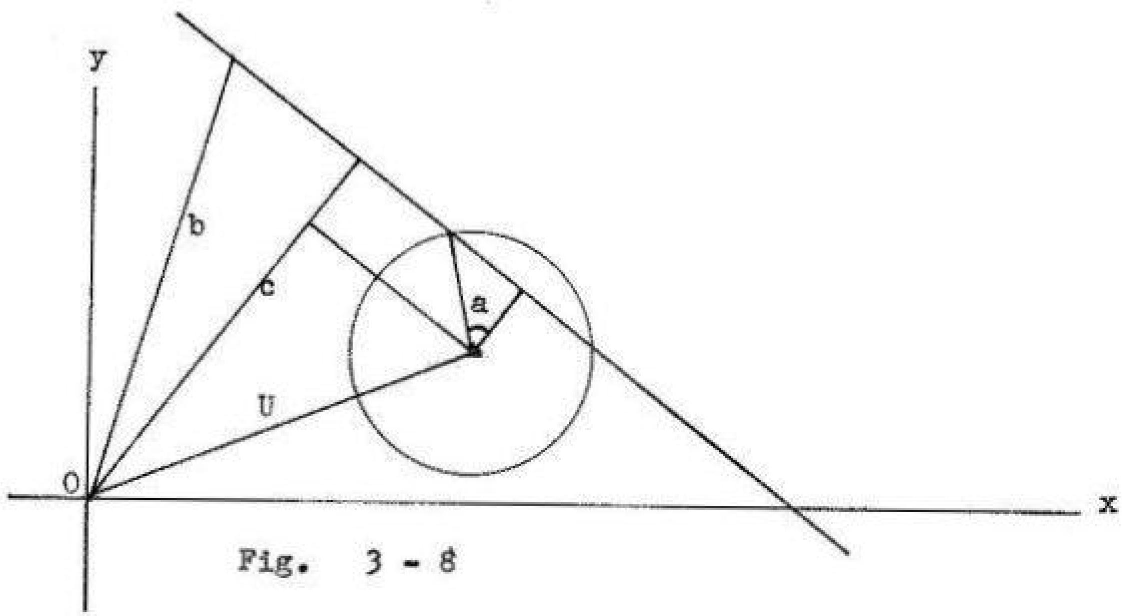


Fig. 3 - 8

Find the equation of a line passing thru a given point and cutting a given circle at a given angle.

Let U be the vector to the center of the given circle of radius a. Let b be the vector to the given point. Let c be the unit normal to the required line. From the figure we may the following equation:

$$(1) \qquad c \cdot U + a e = c \cdot b$$

where e is the cos of the cutting angle. This may be written:

(2) $d \cdot c = a\,e$

(3) $d = b - U$

Solving (2) we obtain :

(4) $c = (a\,e\,d \pm \breve{d}\sqrt{d^2 - (ae)^2})\,d^{-2}$

The equations of the lines may now be written:

(5) $c \cdot r = c \cdot b$

There will be two equations, one for each value of c . The different values of c being due to the two signs in the solution of the proto-type.

When e is unity we get the two tangents thru the point b. It is to be noticed that we get only one line when d is equal to a e. This is never possible as long as b is outside the circle. It is possible when b is inside the circle and it is a possible state of affairs for cutting lines, not tangent lines. When e is zero the cutting line passes thru the center of the circle and in this case we get only one cutting line a s it should be.

One can accomplish the same result by reducing the size of the given circle then finding the tangents to this reduced circle but it is slightly less instructive to do it that way. Another by-product in doing it this way is to gain some sense and feeling for the modes of attack of Mutation Geometry.

If the lines cutting a given circle at a given angle are required to have a specified direction then we must relax the requirement that our c be unity. In this case we shall let c be the vector distance from the origin to the required line. We shall then get the two values below for the magnitude of this perpendicular distance:

(6) $c_0 = c' \cdot U + a\,e$

(7) $c_0 = c' \cdot U - a\,e$

Then the equa tions of thes lines are:

(8) $c' \cdot r = c' \cdot U + ae$

(9) $c' \cdot r = c' \cdot U - ae$

Example I.

For the first example we do one in tangency. In this case the e is unity in the proto-type solution.

Find the equations of the tangents thru the point b (- 10, 8) to the circle whose equation is:

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

Here $b = - 10 i + 8 j$

$U = 2 i + 3 j$

$d = b - U = - 12 i + 5 j$

$a = 5$

$\sqrt{d^2 - a^2} = \pm 12$

$c = (5 (- 12 i + 5 j) + 12 (- 5 i - 12 j))/169$

$c = - (120 i + 119 j)/ 169$

$c \cdot b = 248/ 169$

Our first equation then becomes:

$$120 x + 119 y = - 248$$

It is easy to show that the distance from the center of the circle (2, 3) to this line is 5 showing that it is tangent to the circle. It also passes thru the point b (- 10, 8) thus fulfilling the requirements laid upon it. To get the second tangent we use a minus 12 instead of the plus 12 in the proto-type solution. Doing this we get:

$$c = 169 j/ 169 = j$$

$$c \cdot b = 8$$

The second tangent then has the equation:

$$y = 8$$

It meets all requirements also. See Fig. 3 - 9 .

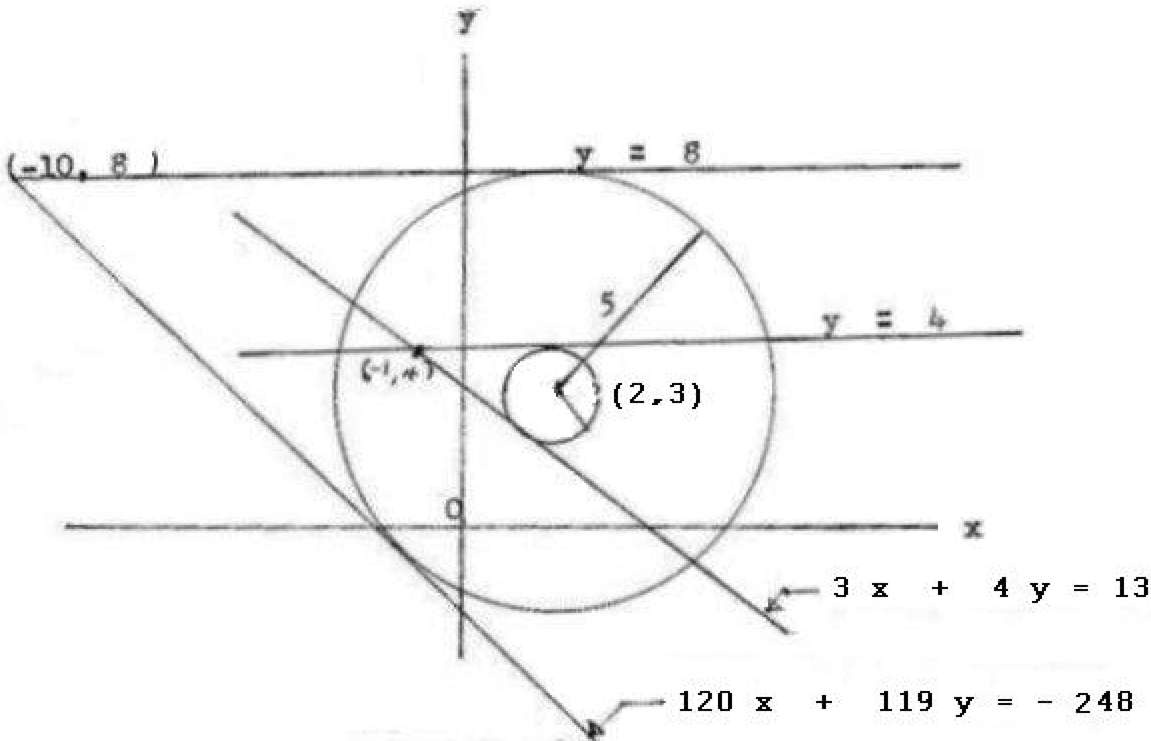


Fig. 3 - 9

The sketch above, in Fig. 3 - 9, depicts the situation for example 1. We shall next illustrate with an example which will deal with a cutting situation.

Example 2.

Find the equations of the lines passing thru the point b (- 1, 4) and cutting at arc cos 1/5 the circle whose equation is:

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

Here
$$U = 2i + 3j$$
$$b = -i + 4j$$
$$d = b - U = -3i + 1j$$
$$s = 1/5$$
$$c = ((-3i + j) \pm (i + 3j)\,3)/10 = -(3i + 4j)/5$$
$$c \cdot b = -13/5\ .$$

Our first cutting equation then becomes:

$$3x + 4y = 13$$

The effective radius of the reduced circle is $s a = 1$ and it is easy to show **that** the line above comes within a distance of 1 of the center of the reduced circle (2, 3). To get the other cutting line we have:

$$c = ((-3i + 1j) + 3(i + 3j))/10$$

$$c = j$$

$$c \cdot b = 4$$

Our second cutting equation then becomes:

$$y = 4$$

It is also easy to show that this line passes at a distance of 1 from the center (2, 3) of the reduced circle thus showing that it is a proper solution. Each equation also passes thru the specified point (- 1, 4). See Fig, 3 - 9 for a view.

Exercises

1 Find the equations of the tangents to the circle $x^2 + y^2 = 9$ which are paralle to the line $3x + 4y = 10$.

2 Find the equations of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $x - 2y = 3$.

3 Find the equations of the tangents to the circle $x^2 + y^2 = 1$ which pass thru the point (1, 3).

4 Find the equations of the lines thru (6, 6) and cutting the circle $(x - 4)^2 + (y - 5)^2 = 4$ at 60 degrees.

5 Determine the angle made by the line joining the points (3, 5) and (1, 4) with the circle $(x - 2)^2 + (y - 3)^2 = 5$

6 Show that the line $3x + 4y = -7$ is tangent to the circle $(x - 2)^2 + (y - 3)^2 = 5^2$

3 - 10 Commom Points of a Given Line and a Given Circle

The equation of the circle may be written :

$$(1) \qquad (r - U)_{0} = a$$

a nd that of the line may be written:

$$(2) \qquad b \cdot r = c$$

Subtracting $b \cdot U$ from each side of (2) our equation becomes:

$$(3) \qquad b \cdot (r - U) = c - b \cdot U = d .$$

Dividing (3) by (1) and we obtain:

$$(4) \qquad b \cdot (r - U)' = d/a = f$$

Solving (4) we obtain :

$$(5) \qquad (r - U)' = (f\,b \pm \breve{b}\sqrt{b^{2} - f^{2}})\,b^{-2}$$

$$(6) \qquad (r - U) = (d\,b \pm \breve{b}\sqrt{a^{2} b^{2} - d^{2}})\,b^{-2}$$

$$(7) \qquad r = U + (d\,b \pm \breve{b}\sqrt{a^{2} b^{2} - d^{2}})\,b^{-2}$$

Equation (7) is the answer sought. There will be two, one or no points according as ab is greater than, equal to or less than d. Equation (7) is a very beautiful expression. To help the student in the new technique we illustrste with the following simple example.

Example 1.

Find the common points of the line and circle below :

$$x + 2y = 13$$

$$(x - 2)^{2} + (y - 3)^{2} = 25$$

Here

$$b = i + 2j$$

$$U = 2i + 3j$$

$$c = 13$$

$$b \cdot U = 8$$

$$c - b \cdot U = 5 = d$$

$$a = 5$$

$$b^2 = 5$$

$$a^2 b^2 = 125$$

$$d^2 = 25$$

$$\sqrt{a^2 b^2 - d^2} = 10$$

$$r = 2i + 3j + (i + 2j) + 2(-2i + j)$$

$$r = -i + 7j$$

$$r = 2i + 3j + (i + 2j) - 2(-2i + j)$$

$$r = 7i + 3j$$

Our two points of intersection then are $(-1, 7)$ and $(7, 3)$.

The student, no doubt, knows how to solve the problem by the older and conventional methods, that is to substitute the value of x from the linear equation into that of the circle and solve the resulting quadratic equation by either completing the square, factoring, or using the quadratic formula, or some other way. He, if he is careful, will obtain the same answer as that above.

Note now that (7) is simply the proto-type solution. It may be seen by actual substitution that the two points satisfy both the line and the circle and thus they are common points or points of intersection.

Exercises

By proto-type solutions find the points of intersection of the pairs of lines and circles below:

1 $\quad x + 3y = -4, \quad (x - 2)^2 + (y - 3)^2 = 25.$

2 $\quad x + y = 2, \quad (x + 1)^2 + (y - 2)^2 = 1$

3	$x + y = 3,$	$(x + 3)^2 + (y - 4)^2 = 9$
4	$x + y = 2,$	$(x + 1)^2 + (y - 1)^2 = 4$
5	$x - y = 2,$	$(x - 5)^2 + (y - 2)^2 = 1$
6	$x + 7y = 3,$	$(x + 1)^2 + (y + 3)^2 = 25$

3 - 11 Common Points of Two Given Circles.

The equations of our two given circles may be written:

$$(1) \qquad (r - P)^2 = a^2$$

$$(2) \qquad (r - S)^2 = b^2$$

The difference between these two equations may be written:

$$(3) \qquad c \cdot r = d$$

$$c = 2(P - S)$$

$$d = P^2 - S^2 - a^2 + b^2$$

Equation (3) is the equation of the Radical Axis of the two circles. A part of this radical axis is the common chord. See Fig. 3 - 10.

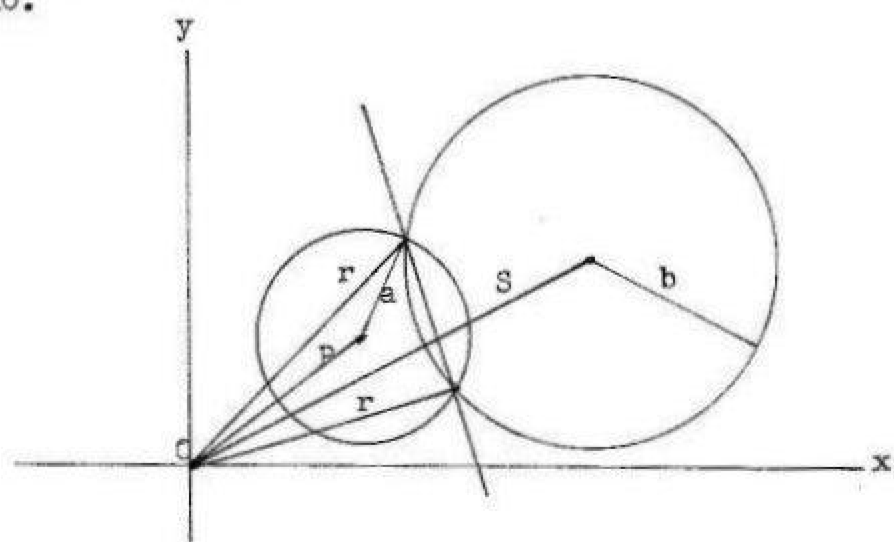


Fig. 3 - 10

Here it has been convenient to use P and S for the vectors from the origin to the centers of the circles whose radii are a and b respectively.

One may now solve the linear equation, the radical axis equation (3) with either equation (1) or (2) and find the common points of the two circles which are the same points as the common points of the radical axis, the common chord, and any one of the circles. This solution will have the same form as equation (7) of section 3 - 10.

The solution in (7) is that of a line and a circle and it is once and for all. One needs never to do it over provided he is satisfied with the correctness of the solution there given and any error in the solution would soon show up in the solution of a number of numerical solutions. None has ever shown up. We accept it as correct. In any situation then one only has to put in the proper values of the constants there given.

If one subtract the two values given in (7), the one from the other, calling their difference g, we obtain:

$$(4)\qquad g = (2\breve{b}\sqrt{a^2 b^2 - d^2})\,b^{-2}$$

$$(5)\qquad g_0 = 2\sqrt{a^2 - d^2 b^{-2}}$$

Here

$$a = a$$

$$b = 2(P - S)$$

$$d = -(P - S)^2 - a^2 + b^2$$

$$U = P$$

For two circles the a,b,d, and U in (7) are to be replaced by their values listed above where the b in d is the radius of one of the circles.

Equation (5) above gives us the length of the common chord. We do an illustrative example:

Example 1.

Find the points of intersection and the length of the common chord of the two circles listed below

$$(x - 2)^2 + (y - 3)^2 = 25$$

$$(x - 0)^2 + (y + 3)^2 = 5$$

Here

$$U = P = 2i + 3j$$
$$S = \quad = 0i - 3j$$
$$P - S = 2i + 6j$$
$$(P - S)^2 = 40$$
$$b = 4i + 12j$$
$$\breve{b} = -12i + 4j$$
$$b^2 = 160$$
$$a^2 = 25$$
$$d = -60$$
$$\sqrt{a^2 b^2 - d^2} = 20$$

$$r = 2i + 3j + \frac{(-60(4i + 12j) + 20(-12i+4j)}{160}$$

$$r = 2i + 3j - 3i - 4j = -i - j$$

For the second value of r we change the sign of the 20 in the above equation and obtain:

$$r = 2i + 3j + 0i - 5j = 2i - 2j.$$

Thus the two points of intersection are $(-1, -1)$ and $(2, -2)$. The distance between these two points is $\sqrt{10}$.

It is easier to take the difference, square and take the square root directly with the arithmetical numbers than to use the distance formula g in equation (5). We confirm our answer $\sqrt{10}$ with that formula.

$$g = 2\sqrt{25 - 3600/160} = 2\sqrt{25 - 45/2}$$
$$= 2\sqrt{5/2} = \sqrt{10}$$

Exercises

1 Find the lengths and equations of the common chords of the following pairs of circles:

1 $(x-2)^2+(y-3)^2=25,\ (x+2)^2+(y+5)^2=25$

2 $(x+1)^2+(y+2)^2=4,\ (x-3)^2+(y-2)^2=20$

3 $(x+2)^2+(y-3)^2=1,\ (x+1)^2+(y-2)^2=1$

4 $(x-4)^2+(y+2)^2=9,\ (x-2)^2+(y+1)^2=8$

3 - 12 Radical Center

It will be shown below, see Fig. 3 - 11, that the radical axes of three circles, taken two at a time pass thru a common point. This point is called the radical center of the three circles.

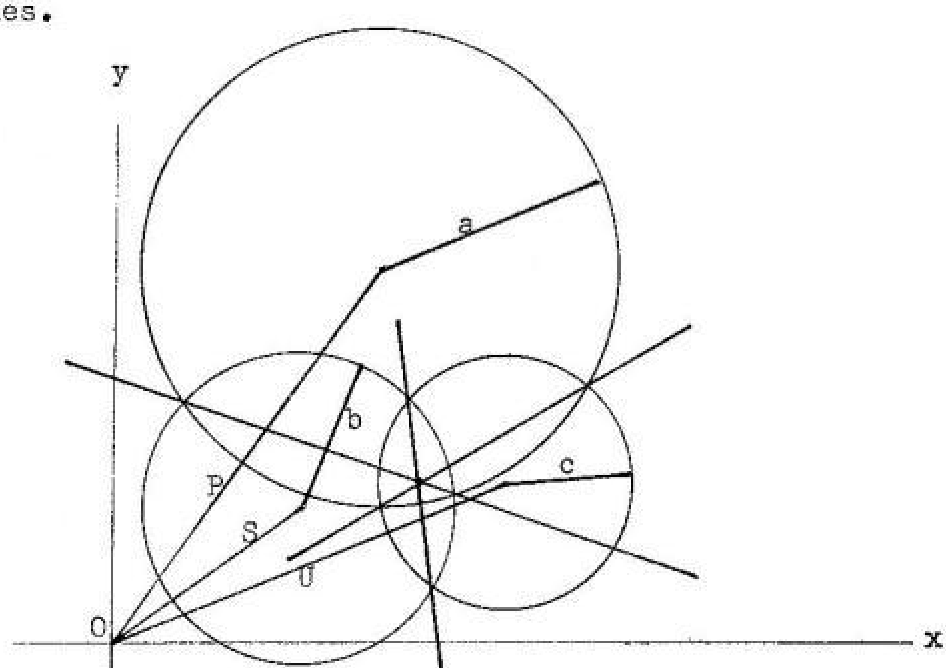


Fig. 3 - 11

Let P, S, and U be the vectore from the origin to the centers of three circles whose radii are a, b, and c respectively. We may then write the three following equations:

$$(1) \qquad (r - P)^2 = a^2$$

$$(2) \qquad (r - S)^2 = b^2$$

$$(3) \qquad (r - U)^2 = c^2$$

From these three equations we obtain the three linear equations:

$$(4) \qquad d \cdot r = L$$

$$(5) \qquad e \cdot r = M$$

$$(6) \qquad f \cdot r = N$$

$$d = 2 (P - S)$$

$$e = 2 (P - U)$$

$$f = 2 (S - U)$$

$$L = P^2 - S^2 - a^2 + b^2$$

$$M = P^2 - U^2 - a^2 + c^2$$

$$N = S^2 - U^2 - b^2 + c^2$$

Solving (4) and (5) a ccording to § (11), 1 - 5 we get:

$$(7) \qquad r = - (M \breve{d} - L \breve{e})/d \cdot \breve{e}$$

Put (7) into the left side of (6) and we get for the left side of (6) the expression:

$$- (M f \cdot \breve{d} - L f \cdot \breve{e}) / (d \cdot \breve{e})$$

By actual multiplication one can find:

$$f \cdot \breve{d} = f . \breve{e} = - d . \breve{e}$$

Thus the left side of (6) becomes: $M - L = N$ which is the right side of (6) . Thus the three lines are concurrent.

Shifting Slurs

In any equation containing **slurred** products it is permissable to shift the slurs from one factor to the other provided the sign of the whole product is changed. At times this is very convenient in simplifying complex expressions. It is one **of** the beautiful facets of Mutation Geometry. In fact we used this facet in establishing the identity:

$$f \cdot \breve{d} = f \cdot \breve{e} = - d \cdot \breve{e}.$$

Equation-wise it may be written in the symbolic form:

$$(8) \qquad \breve{a} \cdot b = - a \cdot \breve{b}$$

This is a very important equation in Mutation Geometry. It is a corollary of the alpha postulate. It flows directly from it. It is one form of the tempo-local invariance required and demanded of the alpha and omega products. For a visual " Proof " see Fig. 3 - 12 .

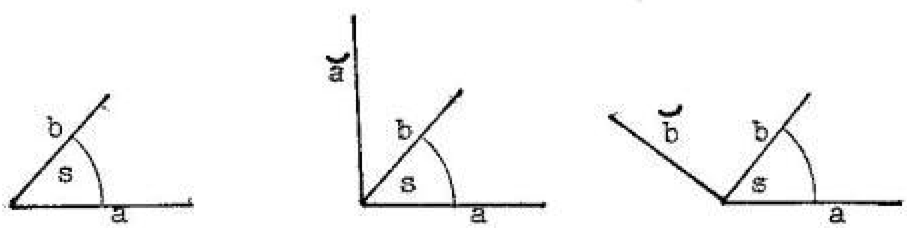


Fig. 3 - 12

The first sketch on the left is that of vectors a and b with angle s between them and its alpha product may be written:

$$a \cdot b = a\, b \cos s$$

The equation for the second sketch may be written:

$$\breve{a} \cdot b = a\, b \cos (\, 90 - s\,) = a\, b \sin s$$

The equation for the third sketch may be written:

$$a \cdot \breve{b} = a\, b \cos (\, 90 + s\,) = - a\, b \sin s$$

If one compares the last two equations it is seen that equation (8) is true.

Example 1.

Find the radical center of the three circles whose equations are listed below.

$$(x-2)^2 + (y-3)^2 = 25$$
$$(x-3)^2 + (y-1)^2 = 40$$
$$(x-1)^2 + (y-0)^2 = 5$$

If we take the difference between the first and second and the first and third we get the two linear equations:

$$x - 2y = -9$$
$$x + 3y = -4$$

the solution of which is the radical center point $(-7, 1)$. Equation (7) gives the same answer. While equation seven is formally correct it should be kept in mind, as we pointed out in the section on system solutions, that one should not make use of the denominators, if one wants to solve a system easily. Any other combination of the three circles will give the same point since we proved that the three chords pass thru the same point.

Exercises

Find the radical center for the following set of triplets of circles;

1

$$(x+1)^2 + (y-4)^2 = 25$$
$$(x-3)^2 + (y-2)^2 = 17$$
$$(x-1)^2 + (y-0)^2 = 5$$

2 $$x^2 + y^2 - 4x - 6y + 8 = 0$$
$$x^2 + y^2 - 2x - 8y + 12 = 0$$
$$x^2 + y^2 - 5x - 7y + 4 = 0$$

3 $$x^2 + y^2 + 3x + 1y - 2 = 0$$
$$x^2 + y^2 + 2x - 1y - 5 = 0$$
$$x^2 + y^2 + 5x - 2y - 10 = 0$$

4 $$x^2 + y^2 + 5x - 2y - 7 = 0$$
$$x^2 + y^2 + 3x - 1y - 3 = 0$$
$$x^2 + y^2 + 2x - 3y + 4 = 0$$

3 - 13 Linear Combination of two Circles

If two circles intersect any other circle thru their two points of intersection may be written as a linear combination of the two given circles. Let the equations of the given circles be written:

(1) $$(r - P)^2 = a^2$$

(2) $$(r - S)^2 = b^2$$

Then any circle thru their points of intersection may be written:

(3) $$k((r - P)^2 - a^2) = (r - S)^2 - b^2$$

where k is a scalar multiplier. This is so for if c is a vector to a common point on both circles it will satisfy both (1) and (2) and thus (3) for it reduces each side to zero.

If c is any other point other than the points of intersection then the circle is required to pass thru three points, the two points of intersection and point c and three conditions determine the circle completely. Substituting c for r in (3) it determines k:

$$(4) \qquad k = ((c - S)^2 - b^2)/((c - P)^2 - a^2) .$$

The student should note how equation (3), for convenience, is written so that in solving for k one does not have to collect a lot of terms before one can divide.

With the k value in (4) put into (3) it may be written in the form:

$$(5) \qquad (r - U)^2 = d^2$$

$$U = (k P - S)/(k - 1)$$
$$d^2 = b^2 + k^2 a^2 + k ((P - S)^2 - a^2 - b^2))/(k - 1)^2 .$$

Note carefully how the k occur in the expressions for U and d . Equation (5) carries a lot of potential. We have already shown how to find the k in (5) when the the required circle is to pass thru the point of intersection of two given circles and a third given point c and of course this is to be done without finding the points of intersection of the two given circles.

There are three very useful results that may be exploited out of equation (5). They are:

Find the equation of a circle passing thru the points of intersection of two given circles and passing thru a specified point.

Find the equation of a circle passing thru the points of intersection of two given circles and cutting a given line at a given angle.

Find the equation of a circle passing thru the points of intersection of two given circles and cutting a given circle at a given angle.

Before we deal with the theory of the later two of these we shall do an illustrative example relating to the first of the three.

Example 1.

Find the equation of a circle passing thru the points of intersection of the two circles whose equations are:

$$(x - 2)^2 + (y - 3)^2 = 25$$
$$(x - 0)^2 + (y + 3)^2 = 5$$

which will pass thru the point $(3, 1)$. Here

$$P = 2 i + 3 j$$
$$S = 0 i - 3 j$$
$$c = 3 i + 1 j$$
$$a^2 = 25$$
$$b^2 = 5$$
$$(c - S)^2 - b^2 = 20$$
$$(c - P)^2 - a^2 = -20$$
$$K = 20/- 20 = - 1$$
$$U = 1 i - 0 j$$
$$d^2 = 5$$

The required circle then becomes:

$$(x - 1)^2 + (y - 0)^2 = 5$$

As a confirmation that this circle is the correct one the points of intersection of the two given circles are $(-1, - 1)$ and $(2, - 2)$ and the given point is $(3, 1)$. All three points are seen to satisfy the derived circle and thus it is the proper one. For a sketch of this example see Fig. 3- 13.

In the section on college geometry, yet to be done, these examples or at least a good number of them will be done from a mechanical standpoint, that is with a compass and ruler. There we shall make certain simplifications in reference to the origin if we have an origin. It is good to know what to expect. We shall see.

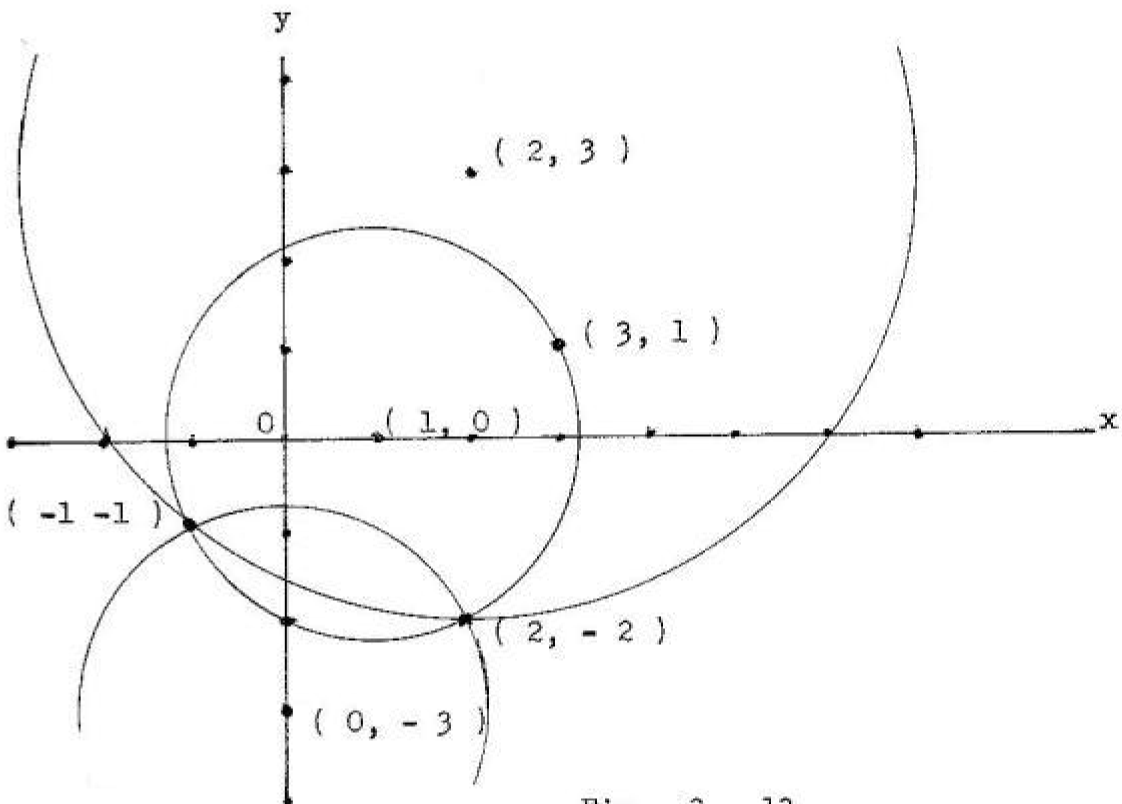


Fig. 3 - 13

The above sketch is for a circle passing thru the points of intersection of two given circles and passing thru a third point.

For the case of a circle passing thru the points of **intersection** of two given circles and making a given angle with a given line we write, as before, for the equations of the two given circles:

$$(6) \qquad (r - P)^2 = a^2$$

$$(7) \qquad (r - S)^2 = b^2$$

and for the required circle we write:

$$(8) \qquad (r - U)^2 = d^2$$

where U and d are given under (5). Let the equation of the given line be.

$$(9) \qquad e' \cdot r = e_0$$

Let the cos of the cutting angle be denoted by f. See Fig. 3 - 14 .

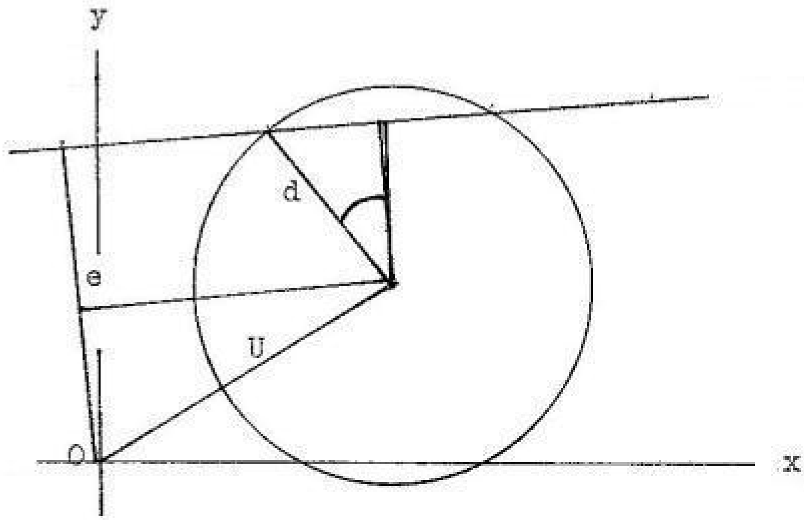


Fig. 3 - 14

From Fig. 3 - 14 we may write the following equation :

$$e' \cdot U + f d = e_0 \qquad (10)$$

Putting in the values of U and d from (5) and solving the resulting quadratic in k one obtains:

$$k = (-M \pm \sqrt{M^2 - 4 L N}) / 2 L$$

$$L = a^2 f^2 - e^2 + 2 e \cdot P - (e' \cdot P)^2$$

$$M = f^2 ((P - S)^2 - a^2 - b^2) + 2 e^2 - 2 e \cdot (P + S) + 2 e' \cdot P e' \cdot S$$

$$N = b^2 f^2 - e^2 + 2 e \cdot S - (e' \cdot S)^2$$

With the value of k computed above inserted, equation (5) gives the required equation; a circle passing thru the points of intersection of two given circles and cutting a given line at a given angle. The amount of work here seems considerable in comparison with that for the same problem done in § 3 - 8 . One must use good judgement in the choice of his methods of attack. It is not dificult to find the two points of intersection of two given circles and then one msy use them as in § 3 - 8. There are many things to consider in making a choice. Elegance is one factor. I like to see my answers come out looking polished. Ease of execution is another factor. Ease and elegance are not always synonomous.

3 - 14 Circle Thru two Points Cutting a Given Circle at a Given Angle.

Find the equation of a circle passing thru two points and cutting a given circle at a given angle. See Fig. 3 - 15

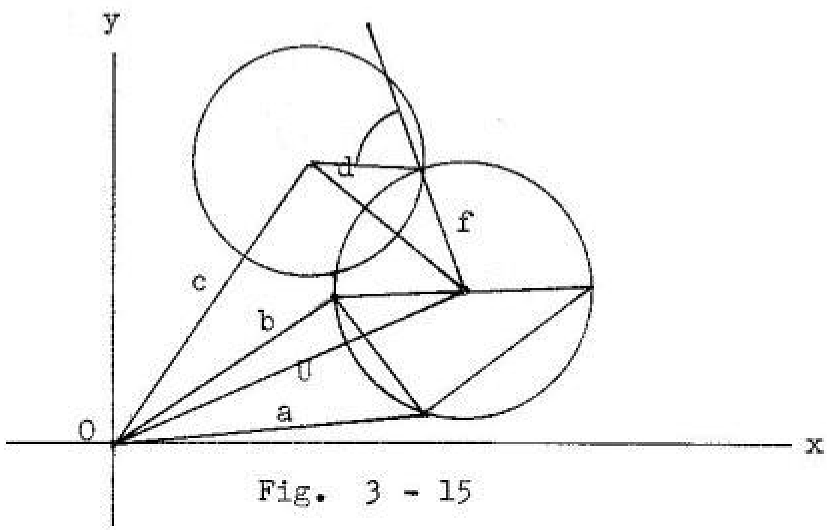


Fig. 3 - 15

Let a and b be the vectors to the given points, U the vector to the center of the required circle. Let c be the vector to the center of the given circle whose radius is denoted by d. Let e be the cos of the cutting angle. Let f be the radius of the required circle. From the figure we may write the following equations:

(1) $\quad (U - b)_0 = f$

(2) $\quad (U - c)^2 = f^2 + 2\, e\, f_0 d + d^2$

(3) $\quad (U - c)^2 = (U - b)^2 + 2\, e\, d\, (U - b)_0 + d^2$

Simplifying (3) we obtain :

(4) $\quad 2\,(b - c) \cdot (U - b) - 2\, e\, d\, (U - b)_0 = d^2 - (c - b)^2.$

If one produces the vector (U - b) until it forms a diameter and projects this diameter on the vector (a - b) one will get the magnitude of (a - b) or equation-wise we may write

(5) $\quad 2\,(a - b) \cdot (U - b) = (a - b)^2 = M$

Dividing (4) by (5) and rearrange and we obtain:

$$(6) \qquad H \cdot (U - b)' = e\,d$$
$$n = (d^2 - (c - b)^2)/(a - b)^2$$
$$H = b - c - n(a - b)$$

Solving the proto-type in (6) we obtain:

$$(7) \qquad (U - b)' = (e\,d\,H \pm \breve{H}\sqrt{H^2 - e^2 d^2})\,H^{-2}$$

Multiplying (7) by 2 (a - b) we obtain:

(8)

$$2(a - b)\cdot(U - b)' = 2(a - b)'\cdot(e\,d\,H \pm \breve{H}\sqrt{H^2 - e^2 d^2})\,H^{-2}$$
$$= N$$

Divide (5) by (8) and we get:

$$(9) \qquad (U - b)_0 = M/N$$

Multiply (7) by (9) , getting (U - b) then transposing the b and we obtain the vector U to the center of the required circle,

$$(10) \qquad U = (M\,H^{-2}/N)\,(e\,d\,H \pm \breve{H}\sqrt{H^2 - e^2 d^2}) + b$$

The required equation may now be written:

$$(11) \qquad (x - h)^2 + (y - k)^2 = (M/N)^2$$

There will be two of these circles; one for each sign in the solution of the proto-type.

We point out that there is no necessity to give coordinates to a, b, and c. If one does not need them then every line in the solution in equation (10) may be constructed with ruler and compass.

In this case, this problem will be recognized as a generalization of one of a sequence of steps in the old Apollonian problem of construction a circle tangent to three given circles.

Emancipation

The student, up to this point, no doubt has been struck by the total absence of any inter-dependence of one proposition on any other. This is a remarkable fact in the nature of the New Science of Mutation Geometry. No sequence, logical or illogical, is present. How could there be any need for any sequence when there is only ONE proposition in the whole geometry.

Creative geometry and the progressive march of civilization have always gone hand in hand. After the Greeks geometry seems to have taken a dive thru the dark ages. In the geometric renaissancs some of the more important contributions were:the Principle of duality by Poncelet and Gergonne, The Barycentric Calculus by Mobius, Projective Geometry, perhaps by many, among whom we mention Von Staudt, Desargues, Analytic Geometry by Descartes, and non-Euclidian Geometry by Lobachevski and Boylai. Of all the different forms of conventional geometry, perhaps, one could choose projective as the most general. One may derive the mensurational geometry of the ancient Greeks from projective by laying certain conditions upon its transformations. Relativity is intimately connected with Riemanian geometry, the eliptic form of non-Euclidian geometry. All these conventional geometries depend on logical order for their very existence. They cannot move without it. Mutation Geometry is not bound by any such restriction.

When one goes from say projective to mensurational geometry one has to sort of shift mental gears. There is a break.

When one goes from Mutation Geometry to projective, analytic, or mensurational geometry there is no break . It is the same whether we are dealing with magnitude or relative position of the elements under consideration. We do only one thing: If the item under consideration is in the alpha state we solve the proto-type. If it is in the omega state we bring it to the alpha state by means of the Omega Proposition, the Proposition of Mutation, and then solve the proto-type. The Proposition of Mutation then must appear as something of a Universal Proposition.

If what we say is true then Mutation Geometry would appear as the most general geometry that man has been able to devise. It might be truly called Pan-geometry.

3 - 15 Circle Passing Thru a Given Point Cutting a Given Line and Circle at Given Angles.

Find the equation of a circle passing thru a given point and cutting a given line and a given circle at given angles. See Fig. 3 - 16 .

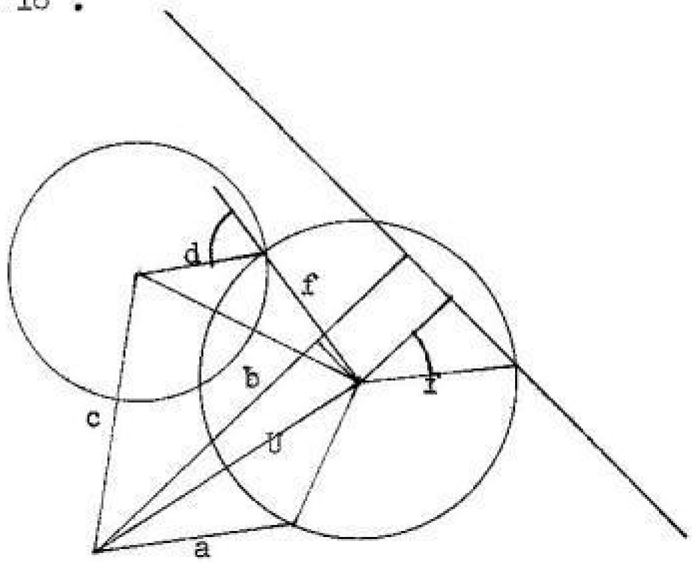


Fig. 3 - 16

Let a be the vector to the given point. Let b be the vector perpendicular distance from the origin to the cutting line. Let c be the vector from the origin to the center of the given circle whose radius is denoted by d. Let e be the cos of the cutting angle between the two circles and g the cos of the cutting angle between the given line and the required circle. Let U be the vector from the origin to the center of the required circle whose radius is denoted by f. From the figure we may write the following equations:

$$(1) \qquad (U - a)_0 = f$$

$$(2) \qquad b' \cdot U + g f = b_0$$

Equation (2) may be written in the form:

$$(3) \qquad b' \cdot (U - a) + g (U - a)_0 = b_0 - b' \cdot a = M$$

$$(4) \qquad (U - c)^2 = f^2 + 2 e d f + d^2.$$

Equation (4) may be written in the form:

$$(5) \qquad 2(a-c)\cdot(U-a)-2ed(U-a)_0 = N$$
$$N = d^2 - (a-c)^2.$$

Dividing (3) by (5) and rearranging we obtain:

$$(6) \qquad H\cdot(U-a)' = G$$
$$n = M/N$$
$$H = 2n(a-c)-b'$$
$$G = g+2ned.$$

Solving the proto-type in equation (6) we obtain:

$$(7) \qquad (U-a)' = (GH \pm \breve{H}\sqrt{H^2 - G^2})\,H^{-2}$$

Multiplying (7) by b' and adding g to the result we obtain:

$$(8) \qquad b'\cdot(U-a)' + g = b'\cdot(GH \pm \breve{H}\sqrt{H^2-G^2})H^{-2} + g = L$$

Divide (8) into (3) and we obtain :

$$(9) \qquad (U-a)_0 = M/L$$

Multiplying (7) by (9) and we obtain,after transposing the a:

$$(10) \qquad U = a + (M/L)(GH \pm \breve{H}\sqrt{H^2-G^2})\,H^{-2}$$

The equation of the required circle may now be written:

$$(11) \qquad (x-h)^2 + (y-k)^2 = (M/L)^2.$$

There will be two circles, one for each sign in the proto-type.

Example 1.

Find the equation of a circle passing thru (- 1, - 1) and cutting the line and circle:

$$x + y = 12$$

$$(x + 2)^2 + (y - 7)^2 = 49$$

at angles:

$$\widehat{\cos}\ 7\sqrt{2}/10$$

$$\widehat{\cos} - 6/10$$

respectively.

Here

$$a = - i - j$$

$$c = - 2 i + 7 j$$

$$e = - 6/10$$

$$g = 7 \sqrt{2}/10$$

$$b' = (i + j)/\sqrt{2}$$

$$b_0 = 12/\sqrt{2} = 6\sqrt{2}$$

$$d = 7$$

$$(a - c) = i - 8 j$$

$$b' \cdot a = - \sqrt{2}$$

$$M = b_0 - b' \cdot a = 7\sqrt{2}$$

$$N = d^2 - (a - c)^2 = - 16$$

$$n = M/N = - 7\sqrt{2} / 16$$

$$H = 2 n (a - c) - b' = \sqrt{2} (- 11 i + 52 j)/8$$

$$G = g + 2 n e d = 35\sqrt{2} / 8$$

$$H_o = 2825/32$$

$$G = 1225/32$$

$$\sqrt{H^2 - G^2} = 5\sqrt{2}$$

$$\breve{H} = -\sqrt{2}\ (52\,i + 11\,j)/8$$

$$G\,H = (-385\,i + 1820\,j)/32$$

$$5\sqrt{2}\ \breve{H} = -(2080\,i + 440\,j)/32$$

$$(G\,H - 5\sqrt{2}\,\breve{H})\,H^{-2} = (3\,i + 4\,j)/5$$

$$L = \overset{\prime}{b}\cdot(G\,H - 5\sqrt{2}\,\breve{H})\,H^{-2} + g = 7\sqrt{2}/5$$

$$M/L = 5$$

$$U = -i - j + (3\,i + 4\,j) = 2\,i + 3\,j$$

and our required circle may be written:

$$(x - 2)^2 + (y - 3)^2 = 5^2$$

The second circle is obtained by changing the sign in the solution of the proto-type. It is

$$(x + 534/41)^2 + (y - 235/41)^2 = (565/41)^2$$

Testing shows that both equations satisfy all the conditions.

The distance from the center (2, 3) of the first circle to the line $x + y = 12$ is $7/\sqrt{2}$ and if we divide this by the radius 5 we get:

$$g = (7/\sqrt{2})/5 = 7\sqrt{2}/10$$

which satisfies. We also have the relation:

$$5^2 + 7^2 + 2(5)(7)\,e = (2 + 2)^2 + (3 - 7)^2$$

whence $\quad e = -6/10$

In identically the same way the other circle may be tested for compliance with the specified conditions.

We point out again that every part in equation (10) may be constructed with straight edge and compass. In that sense it is a problem in college geometry. If we were doing it strictly as a problem in college geometry we would consider every thing from the given point on the required circle which would bring certain simplifications into the final expression. When we come to the section on college geometry we shall re-do a number of the problems here considered from a slightly different viewpoint.

3 - 16 Circle Passing Thru a Given Point Cutting two Given Circles at Given Angles.

Find the equation of a circle passing thru a given point and cutting two given circles at given angles. See Fig. 3 - 17.

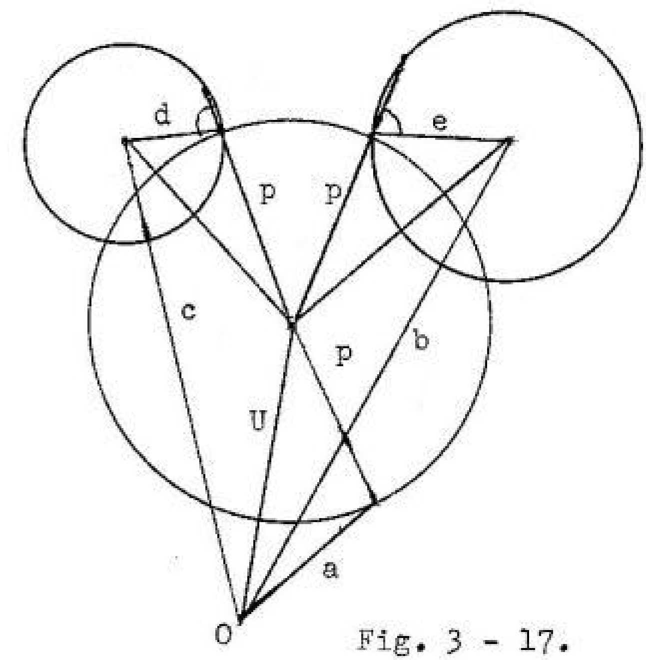


Fig. 3 - 17.

Let a be the vector from the origin to the given point. Let b, e, $\widehat{\cos}$ f, and c, d, $\widehat{\cos}$ g be the vectors to the centers, radii, and cutting angles respectively to the two given circles. Let U be the vector to the center of the required circle whose radius is denoted by p. The following equations then may be written:

$$(1) \qquad (U - a)_0 = p$$

$$(2) \qquad (U - b)^2 = e^2 + 2ef (U - a)_0 + (U - a)^2$$

$$(3) \qquad (U - c)^2 = d^2 + 2dg (U - a)_0 + (U - a)^2$$

From equations (2) and (3), after simplifying, we obtain the following two equations:

$$(4) \qquad 2 (a - b) \cdot (U - a) - 2ef (U - a)_0 = M$$

$$(5) \qquad 2 (a - c) \cdot (U - a) - 2dg (U - a)_0 = N$$

$$M = e^2 - (a - b)^2$$

$$N = d^2 - (a - c)^2$$

Divide (4) by (5) and simplify and we obtain:

$$(6) \qquad H \cdot p' = G$$

$$n = M / N$$

$$H = a - b - n (a - c)$$

$$G = ef - ndg .$$

Solving the proto-type (6) we obtain :

$$(7) \qquad p' = (G H \pm \sqrt{(H^2 - G^2)}\, \breve{H}) H^{-2}$$

$$(8) \qquad 2 (a - b) \cdot p' - 2ef = L$$

$$(9) \qquad (U - a)_0 = M / L$$

$$(10) \qquad U = a + (M / L) (G H \pm \breve{H} \sqrt{H^2 - G^2}) H^{-2}$$

One may now write the equations of the two circles. One is

$$(11) \qquad (x - h)^2 + (y - k)^2 = (M / L)^2$$

The other circle is obtained by changing the sign in the solution of the proto-type.

Example 1.

Find the equation of a circle passing thru (- 1, - 1) and cutting the circles

$$(x - 6)^2 + (y - 5)^2 = 1$$
$$(x + 2)^2 + (y - 7)^2 = 5^2$$

at angles $\widehat{\cos}\, -8 / 10$ and $\widehat{\cos}\, 0$ respectively. **Here**

$$a = -i - j$$
$$b = 6i + 5j$$
$$c = -2i + 7j$$
$$e = 1$$
$$d = 5$$
$$f = -8 / 10$$
$$g = 0$$
$$M = -84$$
$$N = -40$$
$$n = 2.1$$
$$H = -9.1i + 10.8j$$
$$\breve{H} = -(10.8i + 9.1j)$$
$$H^2 = 199.45$$
$$G = -8 / 10$$
$$G^2 = 64 / 100$$
$$\sqrt{H^2 - G^2} = 14.1$$
$$GH = -7.28i + 8.64j$$
$$14.1\breve{H} = -(152.28i + 128.31j$$
$$GH + 14.1\breve{H} = -159.56i - 119.67j$$

One can save time here by using equation (5) instead of (4) in developing the L value since the g term in (5) is zero.

$$2(a - c) = 2(i - 8j).$$

$$2(i - 8j)\cdot(-159.56i - 119.67j) = 1595.6.$$

$$N / 1595.6 = -40 / 1595.6 = -1 / 39.89.$$

$$-(159.56i - 119.67j) / -39.89 = 4i + 3j.$$

$$U = -i - j + (4i + 3j) = 3i + 2j$$

One of the required equations may now be written:

$$(x - 3)^2 + (y - 2)^2 = 5^2$$

The other equation may be obtained by using the alternate sign in the solution of the proto-type. It is left as an exercise for the student. The equation derived above meets all the specified conditions.

3 - 17 A Generalization of the Apollonian Problem

Find the equation of, or construct, a circle which will cut three given circles at given angles. See Fig. 3 - 18 .

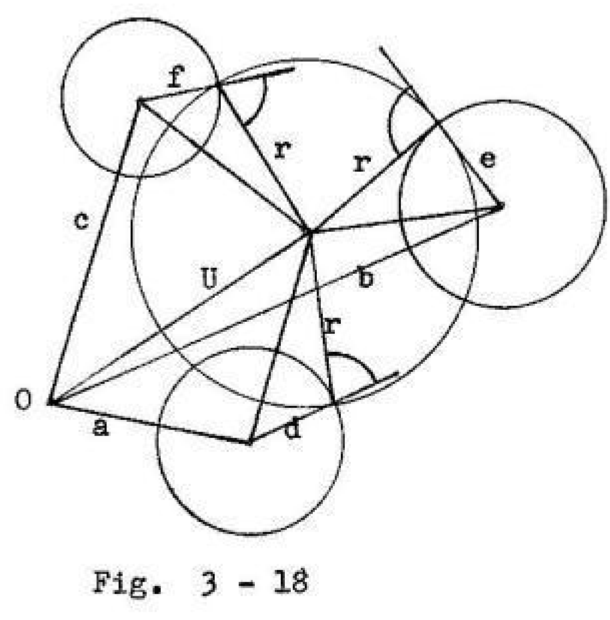


Fig. 3 - 18

Let a, b, c, be the vectors from the origin to the centers of the given circles and d, e, f, be their corresponding radii, and g, p, s, be the corresponding cos of the cutting angles. Let U be the vector to the center of the required circle whose radius is denoted by r. From the diagram the following three equations may be written:

$$(1) \qquad (U - a)^2 = r^2 + 2rdg + d^2$$

$$(2) \qquad (U - b)^2 = r^2 + 2rep + e^2$$

$$(3) \qquad (U - c)^2 = r^2 + 2rfs + f^2$$

By subtracting the first from the second and third we obtain the the following two equations:

$$(4) \qquad A \cdot U = Mr + m$$

$$(5) \qquad B \cdot U = Nr + n$$

$$A = 2(a - b)$$

$$B = 2(a - c)$$

$$M = 2(ep - dg)$$

$$N = 2(fs - dg)$$

$$m = a^2 - b^2 + e^2 - d^2$$

$$n = a^2 - c^2 + f^2 - d^2$$

Solving (4) and (5) for U , according to § (11), 1 - 5 we obtain:

$$(6) \qquad U = Dr + E$$

$$D = (M\breve{B} - N\breve{A}) / A \cdot \breve{B}$$

$$E = (m\breve{B} - n\breve{A}) / A \cdot \breve{B}$$

Put (6) into (1) and one obtains the quadratic in r:

$$(7) \qquad (D^2 - 1)r^2 + 2(D \cdot F - dg)r + F^2 - d^2 = 0$$

$$F = E - a$$

Out of equation (7) one obtains two values of r. Put these two values of r back into equation (6) and one obtains two corresponding values of U. Knowing the center and radius of a circle one may construct it.

Equation (6), a beautiful thing in simplicity, with the aid of equation (7) is a complete generalization of the Problem of Apollonius of Perga.

His Problem was to draw or construct a circle which would be tangent to three given circles. Most solutions of his simple problem,which one can find in most college geometry text books, consists of a series of solutions of still more simple problems, somewhat dependent on each other, until the final solution of the Problem of Apollonius is achieved.

This problem is a historical one. It is attributed to Apollonius of Perga, a Greek geometer. Many geometers have studied this problem since the time of Apollonius who lived from 247-205 BC. The invention of analytic geometry by Descartes did not contribute much toward the generalization or even the solution of the original problem of Apollonius.

It ought to be pointed out that the generalization derived above of the Problem of Apollonius does not depend on any series of dependent solutions. Such ideas and procedures are entirely foreign to Mutation Geometry. Order and sequence, logical or illogical, play no part in its power. In its full scope and sweep of the problems of the geometric world it does not in any sense depend upon order. This is an immense advantage for one can attack problems as he finds them without any reference to any relations that might exist between the one under consideration and any other problem in the geometric world.

In the section of college geometry we shall make an actual construction with ruler and compass for th is generalization which was developed above. One does not know what Apollonius would think of it if he could see it but if one might be allowed to suppose that he would believe in progress it would seem that he would beam some surprise at Mutations Geometry's power over his Pet.

Example 1.

Find the equation of a circle which will cut each of the following three circles at an angle whose cos is written after its corresponding circle.

$$(x - 0)^2 + (y - 1)^2 = 2^2, \text{ ---- } 5 / 16$$

$$(x - 8)^2 + (y - 3)^2 = 3^2, \text{ ---- } 1 / 6$$

$$(x - 1)^2 + (y - 9)^2 = 1^2, \text{ ---- } 3 / 8 .$$

In this case

$$a = 0 i + 1 j$$

$$b = 8 i + 3 j$$

$$c = 1 i + 9 j$$

$$d = 2$$

$$e = 3$$

$$f = 1$$

$$g = 5 / 16$$

$$p = 1 / 6$$

$$s = 3 / 8$$

$$A = - 16 i - 4 j$$

$$B = - 2 i - 16 j$$

$$\breve{A} = 4 i - 16 j$$

$$\breve{B} = 16 i - 2 j$$

$$M = - 1 / 4$$

$$N = - 1 / 2$$

$$m = - 67$$

$$n = - 84$$

$$\mathrm{A} \cdot \breve{\mathrm{B}} = -248$$

$$\mathrm{M}\breve{\mathrm{B}} = -4\,i + 1/2\,j$$

$$\mathrm{N}\breve{\mathrm{A}} = -2\,i + 8\,j$$

$$\mathrm{M}\breve{\mathrm{B}} - \mathrm{N}\breve{\mathrm{A}} = -(\,2\,i + 7.5\,j\,)$$

$$\mathrm{D} = (\,2\,i + 7.5\,j\,) / 248$$

$$\mathrm{E} = (\,736\,i + 1210\,j\,) / 248$$

$$\mathrm{F} = (\,736\,i + 962\,j\,) / 248$$

$$\mathrm{D}^2 - 1 = 60.25 /(\,248\,)^2 - 1$$

$$\mathrm{D} \cdot \mathrm{F} - dg = 8687 /(\,248\,)^2 - 5/8$$

$$\mathrm{F}^2 - d^2 = 1467140 /(\,248\,)^2 - 4$$

One root of equation (7), with the above values inserted, is

$$r = 4$$

Put the values of D, E, and r into equation (6) and one obtains

$$\mathrm{U} = 3\,i + 5\,j$$

One of the required circles may then be written:

$$(\,x - 3\,)^2 + (\,y - 5\,)^2 = 4^2$$

The other value of r and hence U are easy to obtain. The student will have fun doing the other half of the problem. By test the equation above meets all the prescribed conditions.

With this generalization of the historical Problem of Apollonius and the correlated example we bring to a temporary close the Mutation theory of the circle. It would have been more proper to have considered the circle in the general theory of the conics which is to follow but we have followed tradition here and dealt with the circle separately. We shall have more to say about it in the section on college geometry hence the word temporary above. The circle is just as truly a conic as the parabola, ellipse, hyperbola, or the degenerate forms. One will see this in the general theory of the conics.