CHAPTER 4.

CONICS

4 - 1 Conventional View

In the first part of this chapter we shall develope the theory of the conics from the conventional viewpoint with slight changes here and there in the mode and order of presentation with the hope of accelerating the student's rate of absorbing the material and of making it an enjoyable go for the student.

We present this conventional view of analytics, and we hope to do a masterful job of it, because we want the student to compare the old analytics at its best with the New Science of Mutation Geometry. We shall come back over the same material, looked at from the Mutation Viewpoint.

The Ellipse

The ellipse is defined as the locus of a point the sum of whose distances from two fixed points is a postive constant. The fixed points are called the foci of the ellipse. See Fig. 4 - 1 .

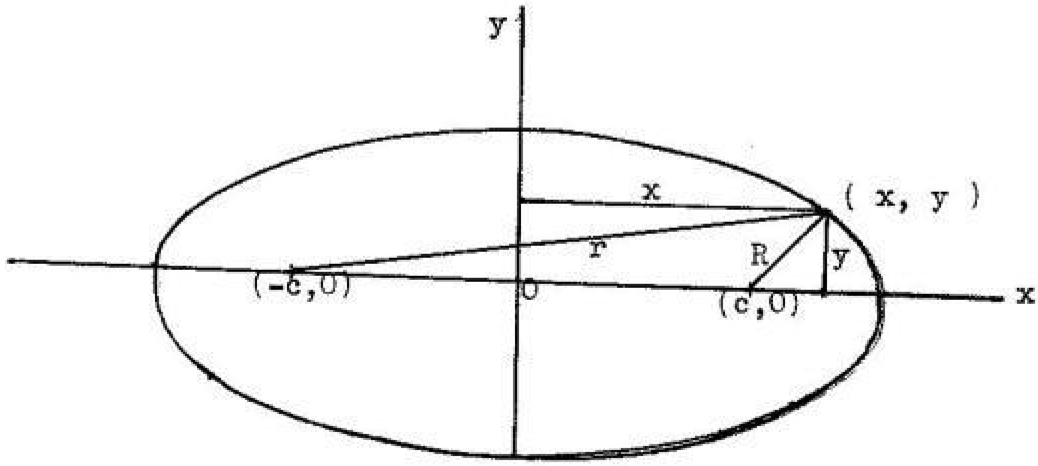


Fig. 4 - 1

Let 2 c be the distance between the two fixed points and 2 a the constant sum. Let the origin be the mid-point of the line joining the two fixed points and the x axis lie along this line. Then the coordinates of the two foci are (- c, 0) and (c, 0).

The following equations may then be written:

$$(1)\qquad r + R = 2a$$

$$(2)\qquad r = ((x+c)^2 + y^2)^{1/2}$$

$$(3)\qquad R = ((x-c)^2 + y^2)^{1/2}$$

Put (2) and (3) into (1) and remove the radicals by squaring twice,and one gets:

$$(4)\qquad x^2/a^2 + y^2/b^2 = 1$$

$$(5)\quad \text{where}\quad b^2 = a^2 - c^2$$

Equation (4), the equation of the elipse, shows what Fig. 4-1 shows; namely, the elipse is symmetric with respect to both the x and y axes. It also shows that the limits of the curve in the x direction are a and - a. Those in the y direction being b and - b. Equation (4) shows that a and b are the semi-major and minor axes of the elipse. When the foci are at the points (0,c) and (0.- c) the equation of the conic becomes:

$$(6)\qquad x^2/b^2 + y^2/a^2 = 1$$

4 - 2 Eccentricity

The quantity c/a is called the eccentricity of the elipse. It is generally designated by the letter e.

Example 1.

Find the semi-major and minor axes and the eccentricity of the following elipse

$$x^2/9 + y^2/4 = 1$$

Here we find $a = 3$, $b = 2$ and $e = c/a = \sqrt{5}/3$.

Exercises

In the following elipses find the semi-axes, major and minor, and the eccentricity.

1 $x^2/4 + y^2/9 = 1$

2 $x^2/1 + y^2/4 = 1$

3 $3x^2 + 2y^2 = 6$

4 $4x^2 + 1y^2 = 36$

5 $1x^2 + 1y^2 = 9$

6 Find the equation of the locus of a point the sum of whose distances from $(\pm 3, 0)$ is 12.

7 Find the equation of the locus of a point the sum of whose distances from $(0, \pm 3)$ is 12.

Perfolatum

(latus rectum)

The perfolatum of any conic is the length of the chord thru the focus perpendicular to the major axis.

The old word for it was the one in the parenthesis above, a slightly mal-euphonic one. I think the namers were trying to be funny. The new word perfolatum is coined from three Latin words meaning thru-the focus-width. It is descriptive and even euphonic. The new word perfolatum maintains the dignity of the English language and will be used hereafter.

If in the standard equation of the ellipse $a^{-2}x^2 + b^{-2}y^2 = 1$ one substitutes c for x and solves for y one should obtain the half value of the perfolatum. It is

$$y = b^2/a = k.$$

The semi-perfolatum will thus be denoted by k. This is for all conics There will occasion no confusion in using k for the semi-perfolatum and k for one of the coordinates of the center of a circle and k for other purposes. If one gets confused over that he has started to study analytics too early.

4 - 3 Standard Forms

The equation of a circle, when its center is at the point (h, K), is written:

$$(1) \qquad (x - h)^2 + (y - k)^2 = r^2.$$

In the same way the equation of an ellipse, when its center is at the point (h, k), is written:

$$(2) \qquad (x - h)^2 / a^2 + (y - k)^2 / b^2 = 1.$$

This last equation will be called the standard form when the origin is not at the center of the ellipse. See Fig. 4 - 2.

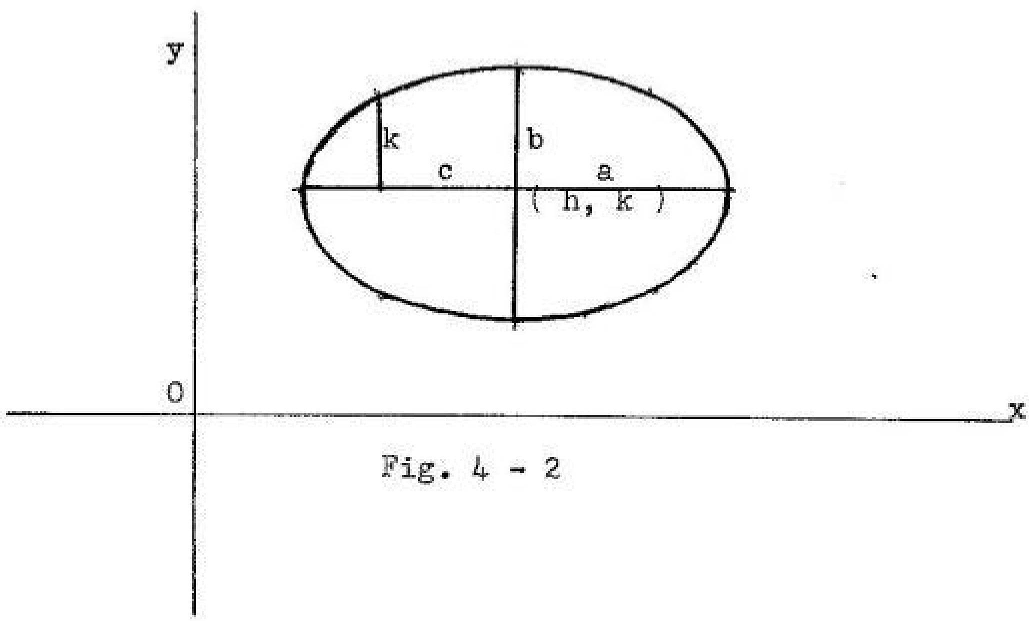


Fig. 4 - 2

If the major axis is in the direction of the y axis the standard equation will be written:

$$(3) \qquad (x - h)^2 / b^2 + (y - k)^2 / a^2 = 1.$$

These two equations, with their major axes along the x and y directions, are the standard forms of the equations of an ellipse.

It is obvious that every equation of the form:

$$(4) \qquad A x^2 + C y^2 + D x + E y + F = 0$$

can, by the completion of squares, be put into the form:

$$(5) \qquad (x + D/2A)^2/(G/A) + (y + E/2C)^2/(G/C) = 1$$

where the quantity G is given by the expression:

$$(6) \qquad G = - F + D / 4 A + E / 4 C .$$

In the statement above the signs of A and C must be alike. They may both be made postive. Equation (5) has the same form as that of equation (2) or (3) and is thus an ellipse.

Example 1.

Put the following equation of an elipse in standard form and find the coordinates of its center, its major and minor axes, its eccentricity, and perfolatum. The equation is:

$$9 x^2 + 4 y^2 - 54 x - 16 y + 61 = 0$$

This may be written in the form

$$(9 x - 54 x + 81) + (4 y - 16 y + 16) = 36.$$

or $$(3 x - 9)^2 + (2 y - 4)^2 = 36$$

or $$(x - 3)^2 / 4 + (y - 2)^2 / 9 = 1 .$$

From this last equation one sees that the center is given by (3, 2). The majorand minor semi-axes are $a = 3$, $b = 2$. The eccentricity is $e = c / a = \sqrt{5} /3$ and the perfolatum $2k = 2 b^2 / a = 8 / 3$.

From the last equation it is also observed that the ma jor axis lies in the direction of the y axis.

Exercises

1 A line segment of length a moves so that its end points are continually on the coordinatesaxes. Show that the locus of any point on this line segment is an elipse. Note the particular type of elipse when the point is the mid-point of the segment. The equation of the elipse is given by $x^2 + n y^2 = (n/(n+1))^2 a^2$ where n is the ratio of the two segment into which the point divides the line segment.

2 Find the center, semiaxes, eccentricity, and perfolatum of the following ellipses:

(a) $x^2 + 4 y^2 - 2 x + 8 y + 1 = 0$

(b) $x^2 + 9 y^2 - 4 x + 36 y + 31 = 0$

(c) $25 x^2 + 16 y^2 - 100 x + 96 y - 156 = 0$

It should be pointed out that when one is given the local of the two foci it is the same as being given its center, and when the center and one semiaxes is known the other semiaxis can be found and thus the conic is known. For instance, if the semi-minor axis of an elipse is 3 and its foci are located at the points (6, 3) and (- 2, 3) then the center of the ellipse is located at (2, 3). Then c = 6 - 2 = 4 and then a = 5. The equation of the ellipse may then be written:

$$(x - 2)^2/ 5^2 + (y - 3)^2/ 3^2 = 1$$

3 Find the equation of an ellipse with semi-major axis 10 and having its foci at the points (- 4, 4) and (8, 4).

The Hyperbola

A hyperbola is the locus of a point the difference of whose distances from two fixed points is a constant. The two fixed points are the foci of the hyperbola. See Fig. 4 - 3.

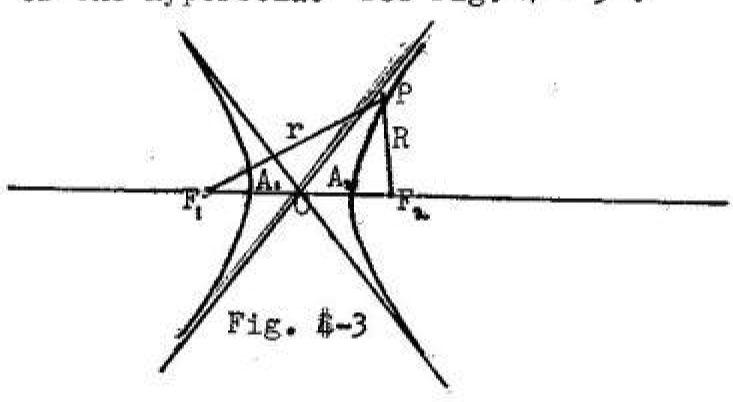


Fig. 4-3

Let F (- c, 0) and F (c, 0) be the foci, A (- a, 0) and A (a, 0) be the vertices of the hyperbola. Let P (x, y) be a point on the hyperbola. Designate P F by r and P F by R. Then from the definition we may write the following equations.

4 - 4 Equation of a Hyperbola

(1) $r_o - R_o = 2a$

(2) $r_o = ((x + c)^2 + y^2)^{1/2}$

(3) $R_o = ((x - c)^2 + y^2)^{1/2}$

Put (2) and (3) into (1) and remove the radicles by squaring and one obtains:

(4) $x^2 / a^2 - y^2 / b^2 = 1$

where b is given by the expression

(5) $b^2 = (c^2 - a^2)$.

We solve the linear equation thru the origin

(6) $y = m x$

with the hyperbolic equation (4) and we obtain:

(7) $x = - a b / (b^2 - m^2 a^2)^{1/2}$.

From (7) one sees that the values of x are unlimited when

(8) $m = \pm b / a$.

Put this value of m into (6) and we obtain the equation

(9) $y = \pm (b/a) x$

Equation (9) is the equation of the asymptotes to the hyperbola.

4 - 5 Equation of a Hyperbola Center not at the Origin.

In analogy with the equations for the elipse The equation of a hyperbola, with center at (h, k) and line of foci parallel to the x axis, may be written:

$$(1) \qquad (x - h)^2 / a^2 - (y - k)^2 / b^2 = 1 .$$

If the line of foci is parallel to the y axis it is written

$$(2) \qquad (x - h)^2 / b^2 - (y - k)^2 / a^2 = 1 .$$

It is obvious that every equation of the form:

$$(3) \qquad A x^2 - C y^2 + D x + E y + F = 0$$

may, by the completion of squares, be made to take the form of either (1) Or (2) above, where both A and C are postive. We illustrate with an example.

Example 1.

Simplify the following hyperbolic equation:

$$9 x^2 - 4 y^2 - 54 x + 16 y + 29 = 0.$$

This may be written in the form:

$$9 (x^2 - 6 x + 9) - 4 (y^2 - 4 y + 4) = 36 .$$

or $$(x - 3)^2 / 4 - (y - 2)^2 / 9 = 1$$

The last equation is that of a hyperbola with its center at(3, 2), and line of foci parallel to the y axis. Its semi-axes are a = 3, and b = 2. Its eccentricity is $e = c / a = \sqrt{13} / 3$ and its perfo-latum is $2k = 2 b^2 / a = 8 / 3$.

Exercises

In the following hyperbolic equations find the center, the semi-axes, the eccentricity, and the semi-perfolatum.

1 $$2x^2 - 1y^2 + 4x + 2y - 1 = 0$$

2 $$2x^2 - 3y^2 - 4x + 18y - 31 = 0$$

3 $$1x^2 - 1y^2 + 4x - 2y + 1 = 0$$

4 - 6 The Parabola

A parabola is the locus of a point whose distances from a fixed point and a fixed line are equal. See Fig. 4 - 4 . The fixed point is called the focus and the fixed line the directrix.

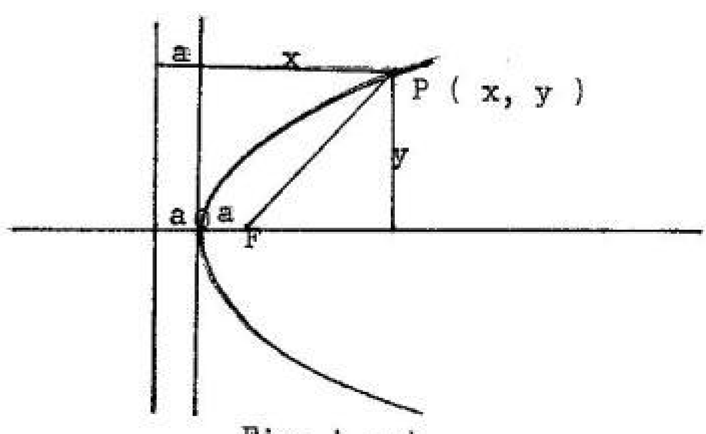


Fig. 4 - 4 .

We shall first consider the case where the vertex of the parabola is at the origin. Let the distance from the focus F to the directrix be denoted by 2 a. Then OF is equal to a. Consider a point P(x,y) on the parabola. The distance from P to the directrix is $a + x$. The distance from P to F is $((x - a)^2 + y^2)^{1/2}$. These two quantities are to be equal. Thus we can write:

$$\sqrt{(x - a)^2 + y^2} = a + x .$$

This reduces to

(1) $$y^2 = 4ax$$

If in equation (1) we substitute (y - k) for y and (x - h) for x we obtain the equation:

$$(2)\qquad (y - k)^2 = 4a(x - h)$$

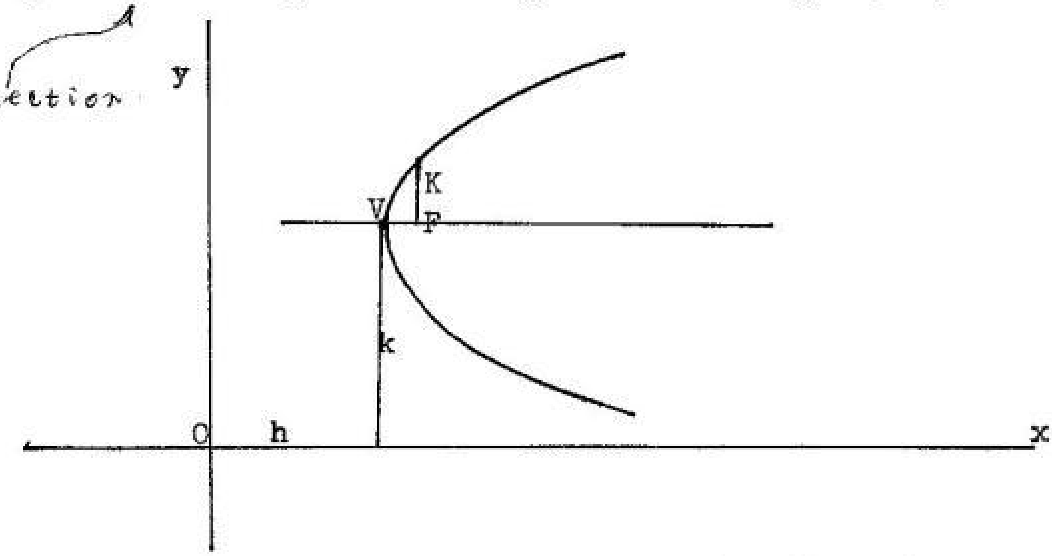
which is the equation of a parabola with its vertex at (h, k) and having its axis parallel to the x axis.

In a similar way we may write the equation:

$$(3)\qquad (x - h)^2 = 4a(y - k)$$

which is the equation of a parabola with its vertex at (h, k) and having its axis parallel to the y axis.

Equations (1), (2), and (3) will open in the postive or negative as a is postive or negative. See Fig. 4 - 5 .



It should be obvious that every equation of the form:

$$(4)\qquad y^2 + Dx + Ey + F = 0$$

can be put into the form of (2) above and that equations of the form:

$$(5)\qquad x^2 + Dx + Ey + F = 0$$

can be put into the form of equation (3) . They thus represent parabolas.

Example 1.

From the following parabolic equation find the vertex, the focus, and the semi-perfolatum:

$$y^2 - 12x - 4y + 40 = 0$$

This may be written in the form:

$$(y - 2)^2 = 12(x - 3) = 4(3)(x - 3).$$

Comparing the last equation with (2) we see that (h, k) = (~~2, 3~~ 3, 2) we also see that a = 3. The focus is then given by (~~5, 3~~ 6, 2).

If in (1) we set x equal to a we obtain :

$$y = K = 2a = 2(3) = 6.$$

In this case we have the semi-perfolatum equal to 6.

It should be observed that the essentials of a parabola are: the local of the vertex and focus, and the length of the perfolatum. The sign of a tells the direction of the open end of the parabola. These values for a parabola can be read from the equations when they are in the form of (2) or (3).

I Exercises

In the following exercises find the local of the vertex and focus, and the length of the semi-perfolatum.

1 $y^2 - 3x - 2y + 7 = 0$

2 $x^2 - 4x - 2y - 4 = 0$

All the conics thus far treated have had their axes parallel to the coordinate axes. This is not always the case. We shall want to deal with them when their axes are initially tilted to the coordinate axes. One cannot change the shape or properties of a conic by the manner in which it is placed on the coordinate axes.

4 - 7 Frame Change, Shift of Viewpoint

Suppose, initially, that our conic is tilted to the coordinate axes. See Fig. 4 - 6 . It is obvious from the figure that we may twist (rotate) our axes around about the origin O until the new x axis (indicated by dashes) becomes parallel to the axis of the conic. If we now view the conic from the new,dashed, axes the fig. will look like Fig. 4 - 2 and the equation corresponding to this fig. had no x y term in it. If the equation, whatever it was, corresponding to our present conic had an x y term in it and it had no x y term after the twist on would come to the conclusion that the twist eliminated the term containing the x y. This is a correct conclusion and we shall show how to find the new equation, referred to the dashed axes, of the conic from the original equation.

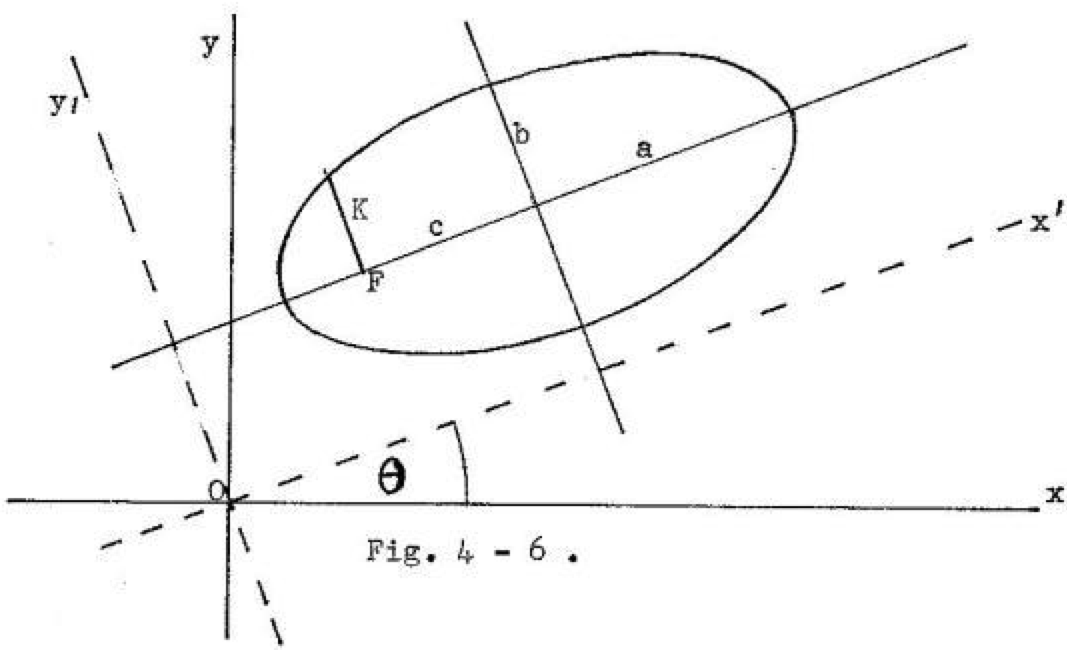


Fig. 4 - 6 .

The most general equation of the second degree may be written

$$(1) \qquad A x^2 + B x y + C y^2 + D x + E y + F = 0 .$$

We shall suppose that (1) represents the conic referred to the original axes. We may write

$$(2) \qquad A' x'^2 + B' x' y' + C' y'^2 + D' x' + E' y' + F' = 0$$

for the equation of the same conic referred to the dashed axes.

Let P (x, y), see Fig. 4 - 7, be a point on a conic, not shown in the Fig, referred to the original axes and P (x', y') the same point on the same conic reffered to dashed axes tilted at an angle θ to the original axes.

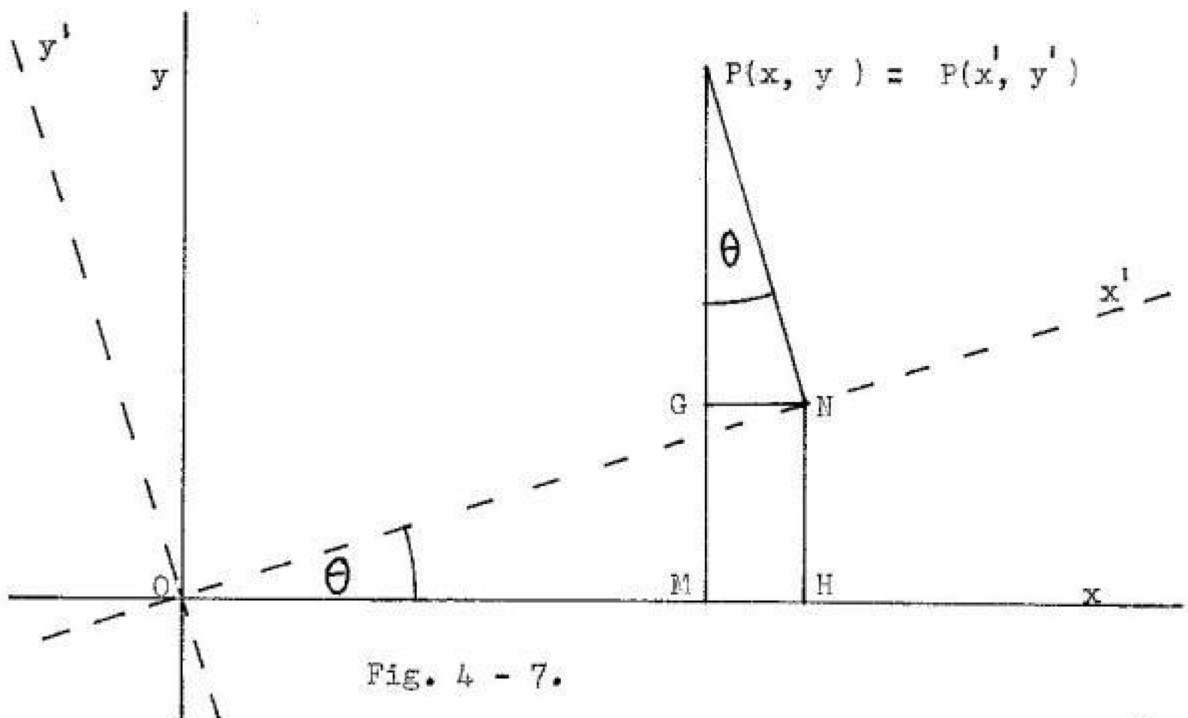


Fig. 4 - 7.

Draw P M perpendicular to Ox and P N perpendicular to O x'. Draw N H perpendicular to O x and N G perpendicular to P M. From the Fig. we see that

$$OM = x, \quad ON = x', \quad PN = y'$$

$$PM = PG + GM = y$$

$$OM = OH - HM = OH - NG = x$$

$$OH = ON \cos\theta = x' \cos\theta$$

$$NG = PN \sin\theta = y' \sin\theta$$

(3) $$x = x' \cos\theta - y' \sin\theta$$

$$P G = P N \cos\theta = y' \cos\theta$$

$$G M = N H = O N \sin\theta = x' \sin\theta$$

(4) $$y = x' \sin\theta + y' \cos\theta$$

Put the values of x and y from equations (3) and (4) into (1), collecting terms and comparing the coefficients of the resulting equation with the coefficients of equation (2) we obtain the resulting set of equations:

(5) $$A' = A \cos^2\theta + B \sin\theta \cos\theta + C \sin^2\theta$$

(6) $$B' = B \cos 2\theta + (C - A) \sin 2\theta$$

(7) $$C' = A \sin^2\theta - B \sin\theta \cos\theta + C \cos^2\theta .$$

Equation (2) was to have no x y term and for this to be true we see from equation (6) that we must have $B' = 0$ or what is the same thing:

(8) $$\tan 2\theta = B / (A - C) .$$

All quantities on the right of (8) are known and it tells us thru what angle we must rotate our original axes in order that our new equation (2) will have no x y term. Twist is a more descriptive word than rotate. When B' is zero equation (2) becomes:

(9) $$A' x'^2 + C' y'^2 + D' x' + E' y' + F' = 0 .$$

It was shown previously that equations of the type of (9) represent an elipse or hyperbola according as A' and C' have like or unlike signs. If, in addition, either A' or C' is zero then (9) represents a parabola.

4 - 8 Conic Types From Invariance

By adding equations (5) and (7) one obtains :

(1) $$A' + C' = A + C$$

If from equations (5), (6), and (7) we form the expression $B'^2 - 4A'C'$ in terms of the right hand members we obtain the result:

$$B'^2 - 4A'C' = (B^2 - 4AC)(\sin^2\theta + \cos^2\theta)^2$$

or

$$(2) \qquad B'^2 - 4A'C' = B^2 - 4AC .$$

Since B' is to be zero this last equation becomes:

$$(3) \qquad -4A'C' = B^2 - 4AC .$$

When A' and C' have like signs, the condition for an ellipse, the right side of (3) is negative and this requires that

$$(4) \qquad B^2 < 4AC . \qquad (\text{ellipse}) .$$

When either A' or C' is zero, the condition for a parabola, the right hand side of (3) is zero and this requires that

$$(5) \qquad B^2 = 4AC . \qquad (\text{parabola}) .$$

When A' and C' have unlike signs, the condition for a hyperbola, the right hand side of (3) is postive and this requires that

$$(6) \qquad B^2 > 4AC . \qquad (\text{hyperbola}) .$$

Notice that the linear term coefficients of an equation representing a conic never occur in the expressions representing the conic types. One ought to be able to tell the type of a conic by just looking at its equation. We look at an equation :

Example 1.

Find the type of the following equation:

$$2x^2 - 4xy + 5y^2 - 12x - 6y - 42 = 0 .$$

Here $B^2 = 4^2 = 16$ and $4AC = 4(2)(5) = 40$. Here 16 is less than 40 and so the relation

$$B^2 < 4AC$$

is satisfied and the equation represents an ellipse. In like manner the other conic types are determined.

The expressions on each side of equations (1) and (2) are called invariants. They are invariant to a rotation of axes. These expressions are the same when the equation is referred to the old axes and when it is referred to the new axes after the rotation. There is another invariant called the discriminant which has to do with the degeneracy of a conic. We shall have no use for it at this time and will not discuss it here. The new science of Mutation Geometry gives a new and compact expression for it compared with what one usually finds in conventional geometry books.

The first thing to do in simplifying the conic equation

$$(7) \qquad A x^2 + B x y + C y^2 + D x + E y + F = 0$$

is to determine its type or species by one of the forms given above. The second thing to do is to find F' in the equation:

$$(8) \qquad A x'^2 + B x' y' + C y'^2 + F' = 0 .$$

where

$$(9) \qquad n = B^2 - 4AC, \quad h = (2CD - BE)/n$$

$$k = (2AE - BD)/n$$

$$F' = A h^2 + B h k + C k^2 + D h + E k + F .$$

The third step in simplifying a conic equation is to find A' and C' in the equation:

$$(10) \qquad A' x''^2 + C' y''^2 + F' = 0 .$$

These are determined from equations (1) and (3) to be

$$S = \sqrt{B^2 + (A - C)^2}$$

$$(11) \qquad A' = (A + C + S)/2, \quad C' = (A + C - S)/2 .$$

Equation (10) is the simplified form for a conic equation representing a circle, ellipse,,or hyperbola. It should be pointed out that,when $B^2 = 4AC$, the condition for a parabola, equation for n in (9) is zero and so one cannot solve for the h and k since the denominators in their expressions is zero. In (11) it is important to know what sign to give to S. We shall soon look, at this question. In the meantime one may calculate the A' and C' from (5) and (7) of the previous section with the help of eq. (8) of that section. For convenience of computation we re-write equations (5) and (7) of the last section in the form:

$$(12) \qquad A' = (A + C + (A - C)\cos 2\theta + B \sin 2\theta)/2$$

$$(13) \qquad C' = (A + C - (A - C)\cos 2\theta - B \sin 2\theta)/2 .$$

We add below the equations for D', E', and F', due to a rotation of axes, for use when the equation under consideration is a parabola.

$$(14) \qquad D' = D \cos\theta + E \sin\theta$$

$$(15) \qquad E' = -D \sin\theta + E \cos\theta$$

$$(16) \qquad F' = F .$$

Note that D' and E' have single angle functions of θ while A' and C' have double angle functions of θ . From equation (8) of the last section we form a right triangle whose base is (A - C), altitude B, and hypotenuse S. From this triangle we may write:

$$(17) \qquad \tan 2\theta = B/(A - C)$$

$$(18) \qquad \sin 2\theta = B/S, \quad \cos 2\theta = (A - C)/S .$$

Put the two equations in (18) into (12) and (13) and we get the two equations in (11) which do not yet tell us the sign of S.

Equations (12) and (13) will always give the correct values for A' and C' when properly used. We do an illustrative example. Simplify the following conic:

Example 2.

$$2x^2 - 4xy + 5y^2 - 12x - 6y + 42 = 0$$

We have previously determined this conic to be an ellipse. From (9) one gets:

$$h = 6, \quad k = 3, \quad F' = -3$$

From (8) of the last section one obtains

$$\tan 2\theta = 4/3, \quad \sin 2\theta = 4/5. \quad \cos 2\theta = 3/5 .$$

Put these last values into (12) and (13) and one obtains

$$A' = 1, \quad C' = 6$$

Our simplified equation then becomes:

$$1x^2 + 6y^2 = 3 .$$

If we had utilized equation (11) to compute the values of A' and C' one might obtain

$$6x^2 + y = 3$$

which is incorrect. This is the values when S is postive. The correct answer is obtained when S is a negative 5. We shall defer an expla nation of this until we do this section from the Mutation Viewpoint. For the present we simply use equations (12) and (13) as we did above to arrive at the correct answer.

Having derived the correct equation one may now find the various elements of interest. We may write the conic in the usual form:

$$x^2/3 + y^2/(1/2) = 1$$

from which we see that the semi-major and minor axes are:

$$a = \sqrt{3}$$

$$b = \sqrt{1/2} = \sqrt{2} / 2$$

$$c = \sqrt{5/2}$$

$$e = c/a = \sqrt{5/6}$$

$$K = b^2 / a = \sqrt{3} / 6$$

$$(h, k) = (6, 3)$$

This table of values gives about everything that one would want to know about the conic. They are the essential things: the semi-axes, the focal distance from the center, the eccentricity, the perfolatum, and the coordinates of its center. See Fig. 4-8 for a sketch of the conic.

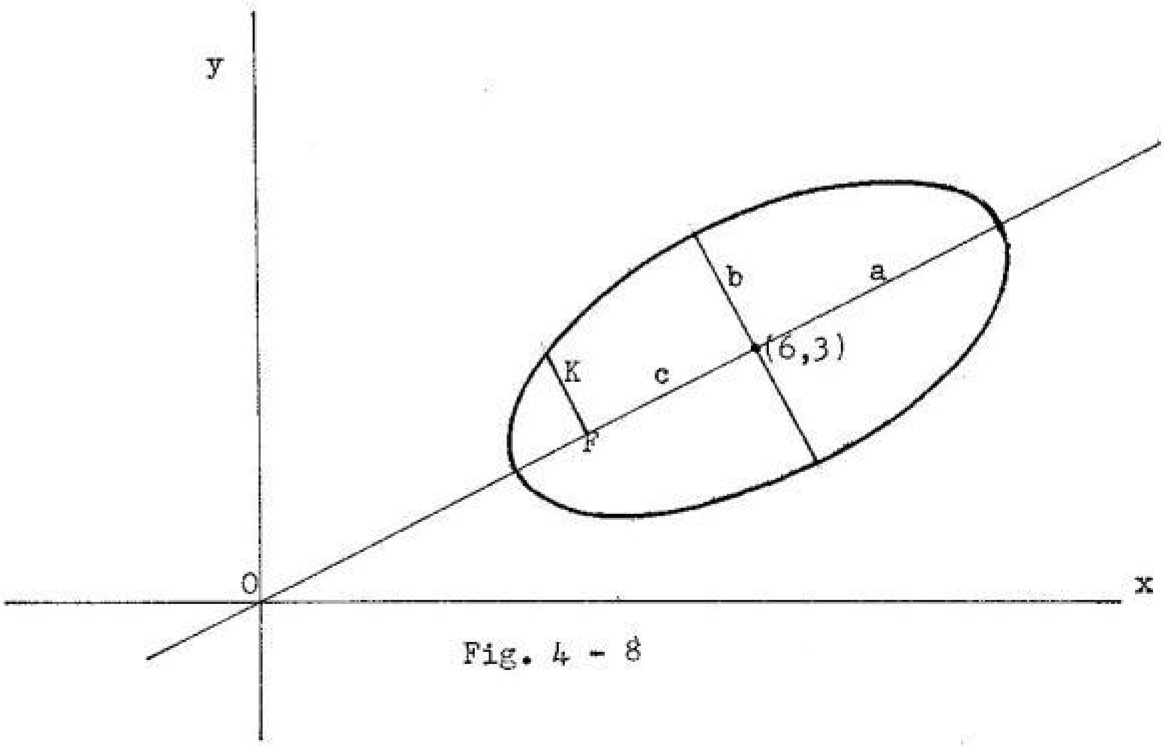


Fig. 4 - 8

To assist the student in grasping a feel for this type of analysis we shall do a second example.

Example 3

Simplify the following conic equation:

$$x^2 + 4xy + y^2 - 2x - 10y - 11 = 0$$

From equation (9) we find :

$$h = 3, \quad k = -1, \quad F' = -9 .$$

From equations (17) and (18) one finds:

$$\tan 2\theta = 4 / 0 = \infty$$

$$\sin 2\theta = 1$$

$$\cos 2\theta = 0 .$$

Put these last two values into equations (12) and (13) and one obtains:

$$A' = 3, \quad C' = -1$$

Our simplified equation then becomes:

$$3x^2 - y^2 = 9 .$$

This is a hyperbola with the following essential values:

$$a = \sqrt{3}, \quad b = 3, \quad c = 2\sqrt{3}, \quad e = 2,$$

$$(h, k) = (3, -1),$$

$$K = b^2/a = 9/\sqrt{3} = 3\sqrt{3}$$

See Fig. 4 - 9 for the local of the hyperbola. It should be observed that one can simplify the conic, both hyperbolas and ellipses, by using only functions of the double angles of rotation. This simplifies the work considerably. In case of a parabola, from the conventional viewpoint, one apparently has to make use of the single angles of rotation and this, at times, can, complicate the work or make the resulting equations slightly awkward or perhaps one would say ugly. May be there are no ugly mathematical equations. It may be that we just do not know how to write them beautifully.

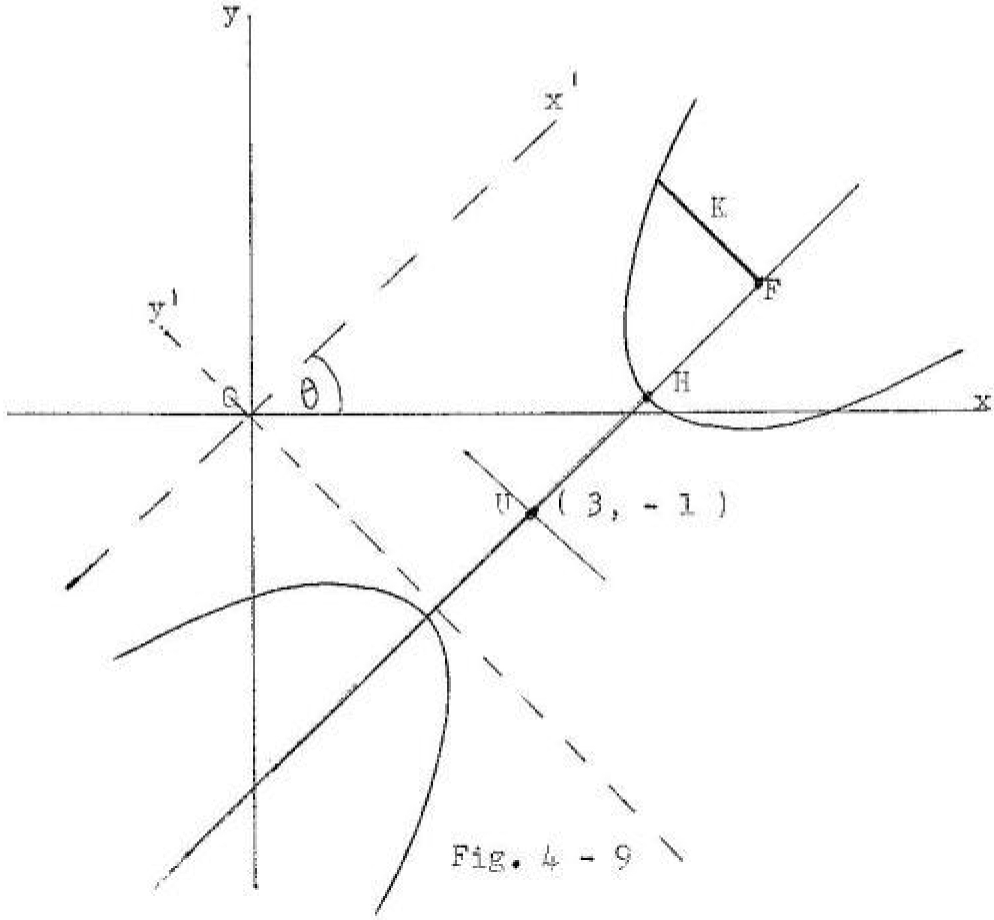


Fig. 4 - 9

To round out this conventional view of the conics we shall look at a parabola which is tilted to its axis of reference.

Example 4.

Find the simplified equation of the conic given by the eq.

$16 x^2 - 24 x y + 9 y^2 - 40 x - 220 y + 700 = 0.$

From this equation we obtain:

$\tan 2\theta = -24/(16-9) = -24/7$

$\cos\theta = 3/5, \quad \sin\theta = 4/5.$

Using the functions in these last two equations we may calculate A', C', D', E', and F' from equations (12), (13), (14), (15), and (16). They are:

$$A' = 0$$

$$C' = 25$$

$$D' = -200$$

$$E' = -100$$

$$F' = 700 .$$

The new transformed equation then becomes:

$$25\,y'^2 - 200\,x' - 100\,y' + 700 = 0 .$$

After division by 25 it may be written in the standard form:

$$(y' - 2)^2 = 8 (x' - 3) = 4 (2) (x' - 3) .$$

If we compare this with the standard equation of a parabola

$$(y' - k)^2 = 4 a (x' - h)$$

we see that its vertex is at the point (3, 2) and that its directional distance a is 2. Its perfolatum is 2 a = 4 . These values are with respect to the rotated axes. It would be a considerable amount of work to compute, say the h and k, values with respect to the original axes. It is more instructive, and revealing, to compute initially all values with respect to the original axes. Mutation Geometry does just that, and without any rotation of axes. We shall yet see. This parabola opens along the postive x axis. See Fig. 4 - 10 for a view of the parabola.

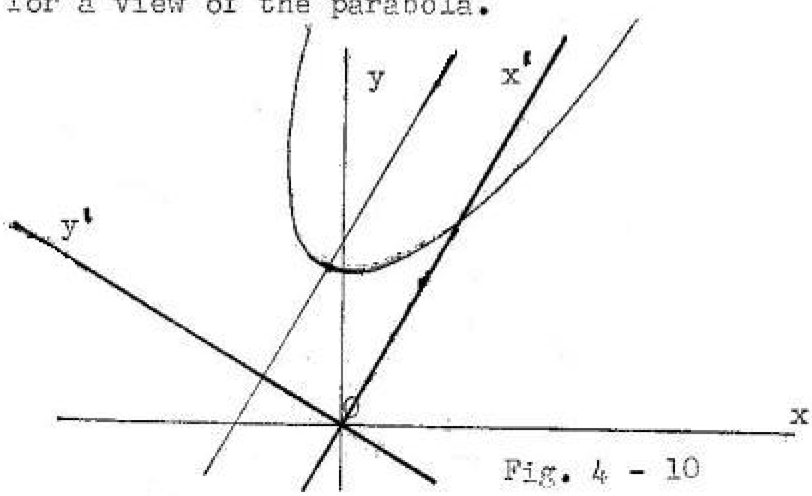


Fig. 4 - 10

Exercises

Determine the species of the following conics then reduce each to a standard form.

1 $\quad 2x^2 + 4xy + 4y^2 + 2x + 4y - 5 = 0$

2 $\quad 3x^2 + 8xy - 1y^2 - 2x - 6y + 4 = 0$

3 $\quad 1x^2 + 2xy + 1y^2 - 1x + 1y - 6 = 0$

4 $\quad 2x^2 + 3xy + 1y^2 + 1x + 1y - 8 = 0$

5 $\quad 1x^2 - 2xy + 1y^2 + 1x + 1y - 4 = 0$

6 $\quad 1x^2 + 1xy + 2y^2 - 1x + 2y - 1 = 0$

7 $\quad xy = 2$.

4 - 9 Polar Coordinates

In rectangular coordinates a point P may be located by P (x, y). this same point may be represented by P (r, θ). See Fig. 4 - 11 .

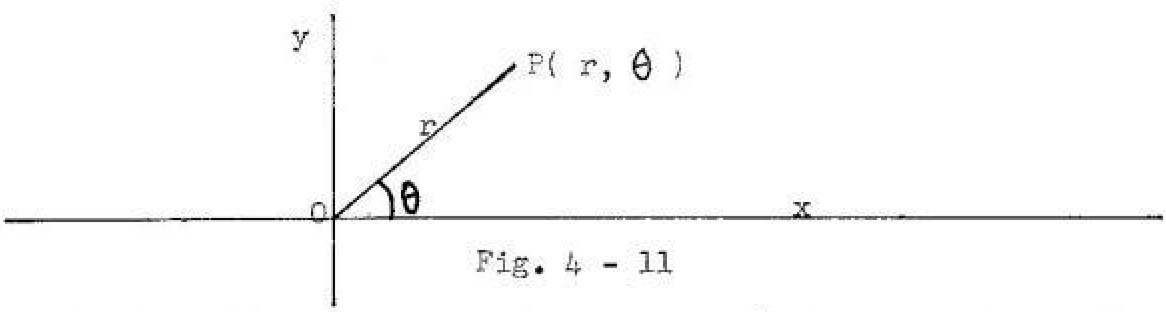


Fig. 4 - 11

Here r is the radius vector to the point and θ is the angle which the radius vector makes with the postive x axis. In this case one may write:

(1) $\quad r = x\,i + y\,j$

(2) $\quad x = r \cdot i = r\cos\theta$

(3) $\quad y = r \cdot j = r\sin\theta$

(4) $\quad r = \sqrt{x^2 + y^2}$

(5) $\quad \theta = \widehat{\tan}\, y/x$.

4 - 10. Polar Equation of a Straight Line .

The rectangular equation of a straight line may be written:

(1) $$A x + B y - C = 0$$

This may be written in the normal form:

(2) $$x \cos S + y \sin S = p$$

where
$$\cos S = A / \sqrt{A^2 + B^2}$$
$$\sin S = B / \sqrt{A^2 + B^2}.$$
$$p = C / \sqrt{A^2 + B^2}.$$

Replacing the x and y in (2) by $r \cos\theta$ and $r \sin\theta$ respectively one obtains the equation:

(3) $$r (\cos S \cos\theta + \sin S \sin\theta) = p$$

This may be written in the form:

(4) $$r \cos (S - \theta) = p .$$

Equation (4) is the equation of a straight line in polar form. See Fig. 4 - 12 for a diagram which could have been used to develope the equation. However, we thought it instructive to show how to go from the equation in one set of coordinates to the equation in another set of coordinates.

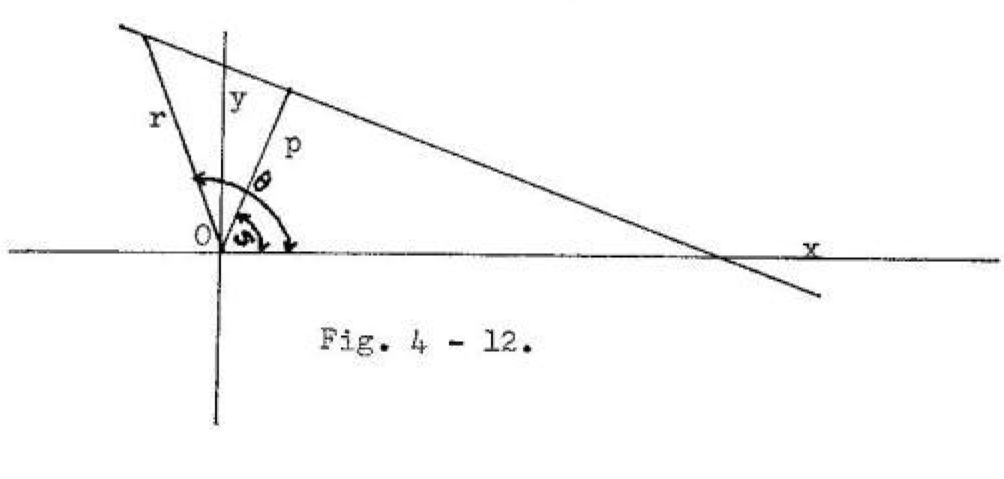


Fig. 4 - 12.

4 - 11 Polar Equation of a Circle

If the center of the circle with radius a is at the origin then obviously its equation is, See Fig 4 - 13.

(1) $r^2 = a^2$

or

(2) $r_0 = a$

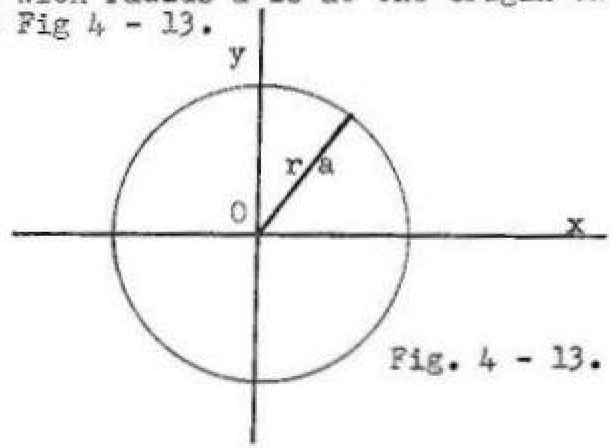


Fig. 4 - 13.

If its center is on the x axis at a distance a from the origin, see Fig. 4 - 14, then its equation is:

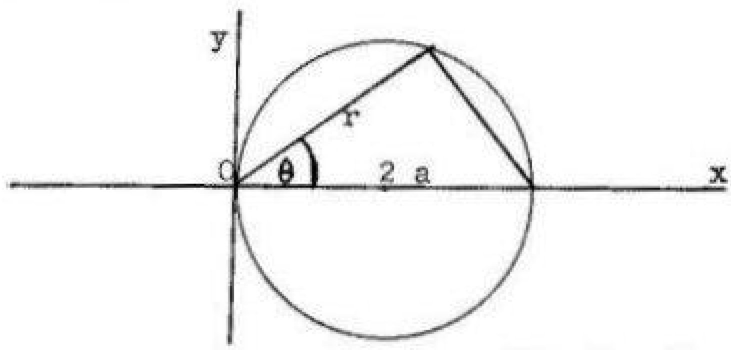


Fig. 4 - 14.

(3) $r = 2a\cos\theta$.

If its center is on the y axis at a distance a from the origin then its equation is

(4) $r = 2a\sin\theta$.

If its center is at the end of vector U then its equation is

(5) $(r - U)^2 = a^2$

or

(6) $(r - U)_0 = a$.

4 - 12. Polar Equation of a Conic

We shall take the origin at the focus. See Fig. 4 - 15.

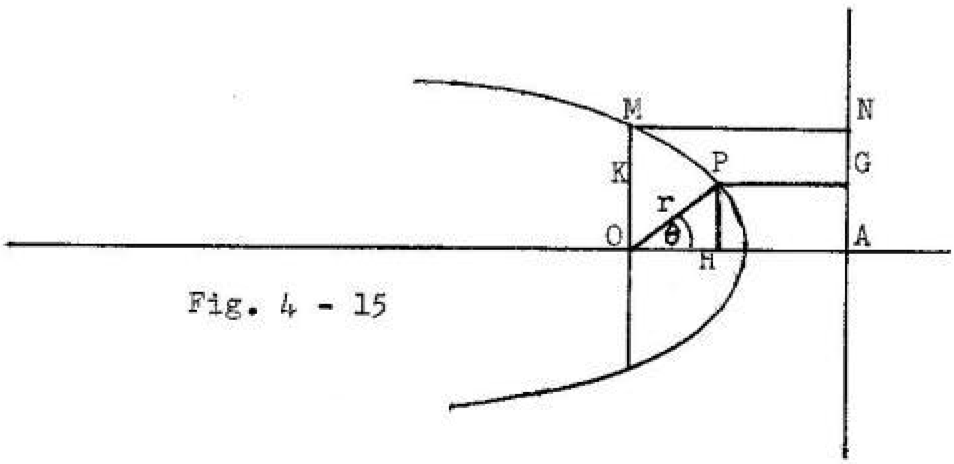


Fig. 4 - 15

Let the line of the foci coincide with the x axis. Let the directrix be denoted by AN which is perpendicular to the x axis. Let OM denote the semi-perfolatum K. Draw MN perpendicular to the directrix. Let P be a point on the conic and let OP be represented by the vector r. Draw PG perpendicular to the directrix. Draw PH perpendicular to the x axis. We may now write the following equation:

(1) $$OH + PG = MN$$

$$OH = r \cos \theta$$

$$PG = r / e$$

$$MN = K / e.$$

Putting these last three equations into equation (1) we obtain

(2) $$r \cos \theta + r / e = K / e .$$

Solving this for r we obtain the polar equation of a conic:

(3) $$r = K / (1 + e \cos).$$

Here e is the eccentricity.