CHAPTER FIVE

Mutation Geometry
View of the Conics

5 - 1. Scalar Sense-tization

Ordinary vectors, like force, velocity, etc, have a magnitude and a direction. Scalars have a magnitude only. If we associate a scalar with a particula r direction we shall say that we have sense-tized the scalar. The eccentricity of a conic is a scalar quantity. It is convenient for us to sense-tize it with the direction of the major axis of the conic under consideration. It will then have components and we may write it:

(1) $$e = e_1 i + e_2 j.$$

(2) $$e^2 = e_1^2 + e_2^2.$$

We shall now find the Mutation Geometry equation of a conic with the origin at one focus. See Fig. 5 - 1.

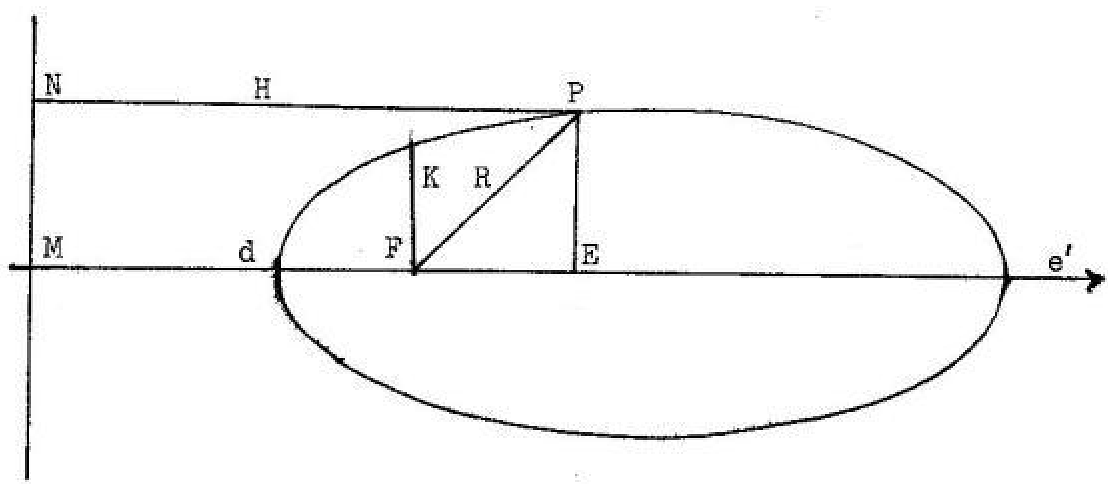

Fig. 5 - 1.

Let F be the focus and F M be the directral distance which we shall designate by d. Let P be a point on the conic. Denote the distance F P by R. Let P N be the perpendicular distance from P to the directal line M N . Designate P N by H. Let the eccentricity have the direction from M to F. By definition the eccentricity is given by the expression:

(3) $$e_o = R_0 / H$$

$$H = MF + FE = d + R \cdot e'$$

Putting this last expression for H into (3) and clearing of fractions one obtains the equation:

(4) $$R_0 = e_0 d + R \cdot e = K + R \cdot e .$$

Here K is the semi-perfolatum. Equation (4) is the Mutation equation of a conic with its focus at the origin and the eccentricity sense-tized along the major axis. We now make the following transformation: (see Fig. 5 - 2)

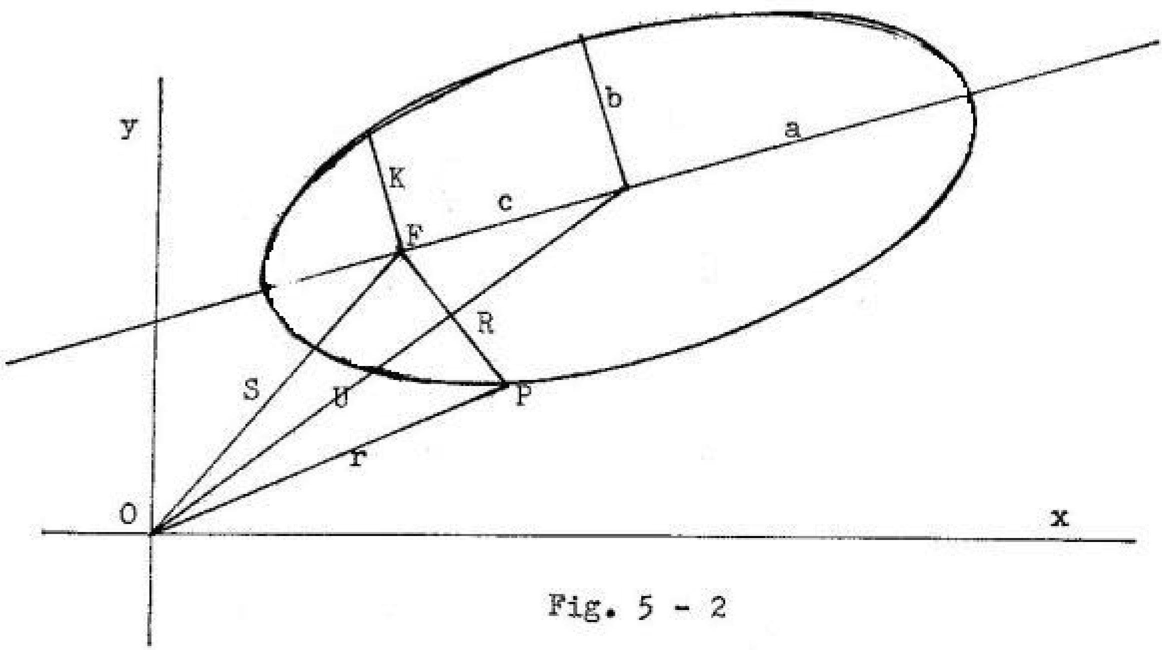

Fig. 5 - 2

(5) $$R = r - S$$

Here S is the vector from the new origin to the focus of the conic. U is the vector from the new origin to the center of the conic and r is the vector from the new origin to any point P on the conic. If we eliminate the R from (4) and (5) we obtain:

(6) $$r^2 - (e \cdot r)^2 = 2 (S + (K - e \cdot S) e) \cdot r + (K - e \cdot S)^2 - S^2 .$$

5 - 2. Primal State Equation

We shall call equation (6) in the last section the Primal State Equation. We now write down the general equation of a conic:

$$A x^2 + B x y + C y^2 = D x + E y + F. \tag{1}$$

We multiply both sides of equation (1) by $2 P^{-1}$ and obtain:

$$a x^2 + b x y + c y^2 = d x + e y + f \tag{2}$$

$$a A^{-1} = b B^{-1} = c C^{-1} = d D^{-1} = e E^{-1} = f F^{-1} = 2 P^{-1}. \tag{3}$$

P is called the Primal State Number of the conic.

We may write (2) in the form:

$$a x^2 + b x y + c y^2 = h . r + f \tag{4}$$

$$h = d i + e j. \tag{5}$$

Here the e in (2) and (5) should not be confused with the eccentricity. In the Primal State Equation (6) of the previous section replace r and the eccentricity e respectively by:

$$r = x i + y j \tag{6}$$

$$e = e_1 i + e_2 j \tag{7}$$

obtaining the equation:

$$\begin{aligned} &(1 - e_1^2) x^2 - 2 e_1 e_2 x y + (1 - e_2^2) y^2 = \\ &2 (S + (K - e . S) e) . r + (K - e . S)^2 - S^2. \end{aligned} \tag{8}$$

We shall identify (4) with (8) exactly. Comparing coefficients we get:

(9) $$1 - e_1^2 = a$$

(10) $$1 - e_2^2 = c$$

(11) $$-2 e_1 e_2 = b$$

(12) $$2 (S + (K - e . S) e) = h$$

(13) $$(K - e . S)^2 - S^2 = f .$$

Eliminating e_1 and e_2 from equations (9), (10), and (11) we get:

(14) $$4 (1 - a) (1 - c) = b^2.$$

Replacing the a, b, and c in equation (14) by their values in (3) we obtain:

(15) $$4 (1 - 2 A P^{-1}) (1 - 2 C P^{-1}) = 4 B^2 P^{-2}.$$

Solving the quadratic in (15) for P we obtain:

(16) $$P = A + C + G_0$$

(17) $$p = A + C - G_0$$

(18) $$G_0 = \sqrt{B^2 + (A - C)^2}$$

(19) $$G = (A - C) i + B j.$$

Thus we have two Primal State Numbers for our conic whose equation is:

(20) $$A x^2 + B x y + C y^2 = D x + E y + F.$$

Our Primal State Equation

(21) $$a x^2 + b x y + c y^2 = d x + e y + f$$

is now known since the P values in equations (16) and (17) are known.

5 - 3. Eccentricity

Adding equations (9) and (10) of the last section we obtain

$$(1) \qquad e^2 = 2 - a - c.$$

If in equation (1) we replace the a and c by $2 A P^{-1}$ and $2 C p^{-1}$ respectively and then replace the P in these expressions by the value of P from equation (16) Of the previous section we obtain the beautiful expression:

$$(2) \qquad e^2 = 2 / \left(1 + (A + C) / \sqrt{B^2 + (A - C)^2} \right)$$

for the eccentricity of a conic in terms of the coefficients of its original equation.

Equation (1) is a historical expression for the eccentricity of a conic. It is expressed in terms of the coefficients of the Primal State Equation of the conic. Take note of how much simpler equation one is compared with equation two. One has only to compute the Primal State Number P to know the Primal State Equation. The Primal State Numbers are just as or perhaps more important than the eccentricity. In truth, it is easy to show that:

$$(3) \qquad e^2 = 1 - p / P = p$$

Equation (3) is another historical expression. In fact, everything that has to do with the Primal State Equation is new to the mind of man.

There had to be a change. It is said that necessity is the mother of invention. In any case the old conventional analytics is too far behind the times. It no longer appeals to the minds and imagination of the student bodies of this land. To continue to rehash the old outmoded analytics to our student bodies is to insult their intelligence.

Example 1.

Write the Primal State Equation and find the eccentricity of the conic whose equation is:

$$2 x^2 - 4 x y + 5 y^2 = 12 x + 6 y - 42.$$

From our given equation we obtain:

$$G_0^2 = (-4)^2 + (2-5)^2 = 16 + 9 = 25$$

$$G_0 = -5$$

$$P = A + C + G_0 = 2 + 5 - 5 = 2$$

$$p = A + C - G_0 = 2 + 5 + 5 = 12.$$

Our primal state number is then to be divided into 2 in order to obtain the proper factor for deriving the Primal State Equatio.

$$2 / p = 2 / 12 = 1 / 6.$$

Multiplying our given equation by $1 / 6$ we obtain:

$$(1/6)(2x^2 - 4xy + 5y^2) =$$
$$(1/6)(12x + 6y - 42)$$

or

$$(1/3)x^2 - (2/3)xy + (5/6)y^2 = 2x + y - 7.$$

This last equation is the Primal State equation of our given eqyation. The eccentricity is given by:

$$e^2 = 2 - a - c = 2 - 1/3 - 5/6 = 5/6$$

$$e = \sqrt{5/6}.$$

Equation (1) for the eccentricity of a conic is unique with the New Science of Mutation Geometry.

We shall now look at some of the other parameters of interest in the conic. It should be pointed out that the Primal State Equation contains the three essential parameters: the eccentricity, which we have just calculated; the focal vector S, and the perfolatum K. We shall presently show how to calculate these in terms of the coefficients of the Primal State Equation.

Before this is done we should like to point out that the eccentricity is beautifully given by equation (3) in terms of the Primal State Numbers of the conic. Using equation (3) we obtain

$$e^2 = 1 \doteq p/P = 1 - 2/12 \doteq 1 - 1/6 = 5/6.$$

It should also be pointed out that equation (2), when cleared of fractions, takes the form:

$$(4) \qquad e^2 = 2\, G_0 / P = p$$

Using this to calculate the eccentricity we obtain the same value:

$$e^2 = 2\,(5)\,/\,12 = 5\,/\,6.$$

I am sure that one is at a loss to choose between the four expressions for the eccentricity. They are all products of the New Science of Mutation Geometry. In them one finds a new sense of direction in geometry. It is only the begin. Having found the Primal State Equation we may solve equations (9) and (10) of $ 5 - 2 for e_1 and e_2 obtaining:

$$(5) \qquad e_1 = \sqrt{1 - a}$$

$$(6) \qquad e_2 = \sqrt{1 - c}.$$

From these two equations we obtain the sense-tized vector eccentricity:

$$((7) \qquad e = i\, e_1 + j\, e_2 = i\sqrt{1 - a} + j\sqrt{1 - c}\ .$$

For the particular example under consideration we have:

$$e_1 = \sqrt{1 - 1/3} = 2/\sqrt{6}$$

$$e_2 = \sqrt{1 - 5/6} = 1/\sqrt{6}$$

$$e = (2i + j)/\sqrt{6}.$$

If we square the last equation we obtain:

$$e^2 = (4 + 1) / 6 = 5 / 6$$

which, in a way, confirms the other expressions for the eccentricity. From equation (7) for the left normal we write:

$$(8) \qquad \breve{e} = - i\sqrt{1 - c} + j\sqrt{1 - a}$$

For the corresponding unit eccentricities we may write from equations (7) and (8) :

$$(9) \qquad \overset{\shortmid}{e} = (i\sqrt{1 - a} + j\sqrt{1 - c}) / e_0$$

$$(10) \qquad \overset{\shortmid}{\breve{e}} = (- i\sqrt{1 - c} + j\sqrt{1 - a}) / e_0$$

It is obvious that equation (7) is slightly more powerful than the other expressions for the eccentricity in that it gives us the sense or direction of the line of foci which is the direction of the major axis. This is important.
For the example under consideration we obtain from equations (8), (9), and (10) the following values:

$$\breve{e} = (- i + 2 j) / \sqrt{6}$$

$$\overset{\shortmid}{e} = (2 i + j) / \sqrt{5}$$

$$\overset{\shortmid}{\breve{e}} = (- i + 2 j) / \sqrt{5} .$$

5 - 4. Focal Vector

We are now in a position to calculate the focal vector S which is the vector from the origin to the focus of the conic. One should expect two vectors. See Fig. 5 - 2 . If we multiply equation (12) # 5 - 2 by $\breve{e}$ we obtain:

$$(1) \qquad \breve{e} . S = h . \breve{e} / 2 .$$

Equation (12) # 5 - 2 may be written in the form:

(2) $$e^2 (K - e \,.\, S)^2 = (h/2 - S)^2$$

Equation (13) # 5 - 2 may be written in the form:

(3) $$e^2 (K - e \,.\, S)^2 = e^2 (S^2 + f) .$$

Comparing the last two equations we obtain:

(4) $$(h/2 - S)^2 = e^2 (S^2 + f) .$$

After expansion and simplifying the last equation may be put it into the following form:

(5) $$(S - h/2(1 - e^2))_0 = e_0\sqrt{h^2 + 4 f (1 - e^2)} \,/\, 2(1 - e^2).$$

Solving (1) and (5) according to (7) # 3 - 10 we obtain:

(6) $$S = U \pm c\, e'$$

Where U is given by the expression

(7) $$U = (h - h \,.\, \breve{e}\, \breve{e}) /2(1 - e^2) .$$

c is given by the expression

(8) $$c = e_0\sqrt{4 f (1 - e^2) + h^2 - (h \,.\, \breve{e})^2} /2(1 - e^2) .$$

It required a slight bit of vision to realize that the expression on the right in (7) should be called U, the vector to the center of the conic and likewise that the expression on the right in (8) should be called c, the distance from the center of the conic to the foci but so they are as designated. One can arrive at the notion in several ways: the best by logic.

If one divides both sides of equation (8) by e_0 the result is:

$$(9) \qquad a = c / e_0 = \sqrt{4 f (1 - e^2) + h^2 - (h \cdot \breve{e})^2} / 2 (1 - e^2).$$

Here a is the semi-major axis. One will not confuse the a and c here with the a and c coefficients in the Primal State Equation. For an ellipse we may now write:

$$b^2 = a^2 - c^2$$

where b is the semi-minor axis, and if one so desires he may write the conic equation in its Primitive Form:

$$x^2 / a^2 + y^2 / b^2 = 1.$$

This, mind, is without a rotation of axes. If it is a hyperbola one may write:

$$b^2 = c^2 - a^2$$

and then the Primitive Equation:

$$x^2 / a^2 - y^2 / b^2 = 1$$

Notice that when the conic is a parabola ($e = 1$) all three of the quantities a, b, and c are infinite. We shall deal with the parabola separately.

When the f, h, and e in equation (9) are replaced by their capitals one may factor out the following signifant results:

$$(10) \qquad a^2 = T / P$$

$$(11) \qquad b^2 = T / p$$

where

$$P = A + C + G_0 \tag{12}$$

$$p = A + C - G_0 \tag{13}$$

$$G = (A - C)\,i + B\,j \tag{14}$$

$$G_0 = \sqrt{(A - C)^2 + B^2} \tag{15}$$

$$n = B^2 - 4AC \tag{16}$$

$$T = 2\left(F + (BDE - AE^2 - CD^2)/n\right). \tag{17}$$

T is called the Transmute Number of the conic. If we replace the a^2 and b^2 in the primitive state equations of a conic by their values in equations (10) and (11) all central conics may now be written in the new Canonical form:

$$P x^2 \pm p y^2 = T \tag{18}$$

The P will be recognized as the Primal State Numbers of the conic.

If the h and e in equation (7), giving the vector U to the center of the conic, are replaced by their caps one obtains the following coordinates to the center of the conic:

$$U_1 = (BE - 2CD)/(B^2 - 4AC) \tag{19}$$

$$U_2 = (BD - 2AE)/(B^2 - 4AC) \tag{20}$$

where our conic is given by:

$$A x^2 + B x y + C y^2 = D x + E y + F. \tag{21}$$

Example 1.

Find the coordinates of the center of the conic:

$$2x^2 - 4xy + 5y^2 = 12x + 6y - 42 .$$

Here

$$A = 2$$
$$B = -4$$
$$C = 5$$
$$D = 12$$
$$E = 6$$
$$F = -42$$

Putting these values into equations (19) and (20) one obtains:

$$U_1 = 6$$
$$U_2 = 3$$

This may be written:

$$(U_1 , U_2) = (6, 3).$$

These are the coordinates of the center referred to its original axes. Putting the values of the coefficients above into equations (12) to (17) 6ne obtains:

$$P = 2$$
$$p = 12$$
$$G = - (3 i + 4 j)$$
$$G_0 = - 5$$
$$T = 6.$$

From equation (18) one may now easily write the Primitive State equation, referred to an axis thru the center of the conic and coinciding with its major and minor axes. It is:

$$x^2 + 6 y^2 = 3$$

It should be pointed out here that in obtaining this Primitive State equation there was no rotation of axes as such.

From equations (10) and (11) or from the last equation one obtains:

$$a = \sqrt{3} = 1.732$$
$$b = \sqrt{2} / 2 = 0.707$$
$$c = \sqrt{a^2 - b^2} = \sqrt{10} / 2 = 1.581$$

From (9) and (10) of # 5 - 3 one has:

$$e' = (2 i + j) / \sqrt{5}$$
$$\overset{\smile}{e} = (- i + 2 j) / \sqrt{5}$$

One may now write for the coordinates of the near focus:

$$S_1 = U - c e' = (6 i + 3 j) - (2 i + j) / \sqrt{2}$$
$$= (4.58 i + 2.29 j).$$

The coordinates of the far focus is given by:

$$S_2 = U + c e' = (6 i + 3 j) + (2 i + j) / \sqrt{2}$$
$$= (7.41 i + 3.71 j).$$

The coordinates of the near vertex may be written:

$$H_1 = U - a e' = (6 i + 3 j) - \sqrt{3} (2 i + j) / \sqrt{5}$$
$$= (4.45 i + 2.23 j).$$

The coordinates of the far vertex may be written:

$$H_2 = U + a e' = (6 i + 3 j) + \sqrt{3} (2 i + j) / \sqrt{5}.$$
$$= (7.54 i + 3.77 j).$$

The Coordinates of the minor near vertex may be written:

$$G_1 = U - b\,\overset{\downarrow}{e} = (6i + 3j) - (-i + 2j)/\sqrt{10}.$$
$$= (6.32i + 2.37j).$$

The coordinates of the minor far vertex may be written :

$$G_2 = U + b\,\overset{\downarrow}{e} = (6i + 3j) + (-i + 2j)/\sqrt{10}.$$
$$= (5.68i + 3.63j).$$

The coordinates of the four points at the ends of the perfolata thru each focus may be written :

$$L_1 = S_1 - K\,\overset{\downarrow}{e}$$
$$L_2 = S_1 + K\,\overset{\downarrow}{e}$$
$$M_1 = S_2 - K\,\overset{\downarrow}{e}$$
$$M_2 = S_2 + K\,\overset{\downarrow}{e}$$
$$K = b^2/a.$$

See Fig. 5 - 3 for a symbolic view of a graph of a conic . It does not numerically represent the calculations above. It is only symbolic.

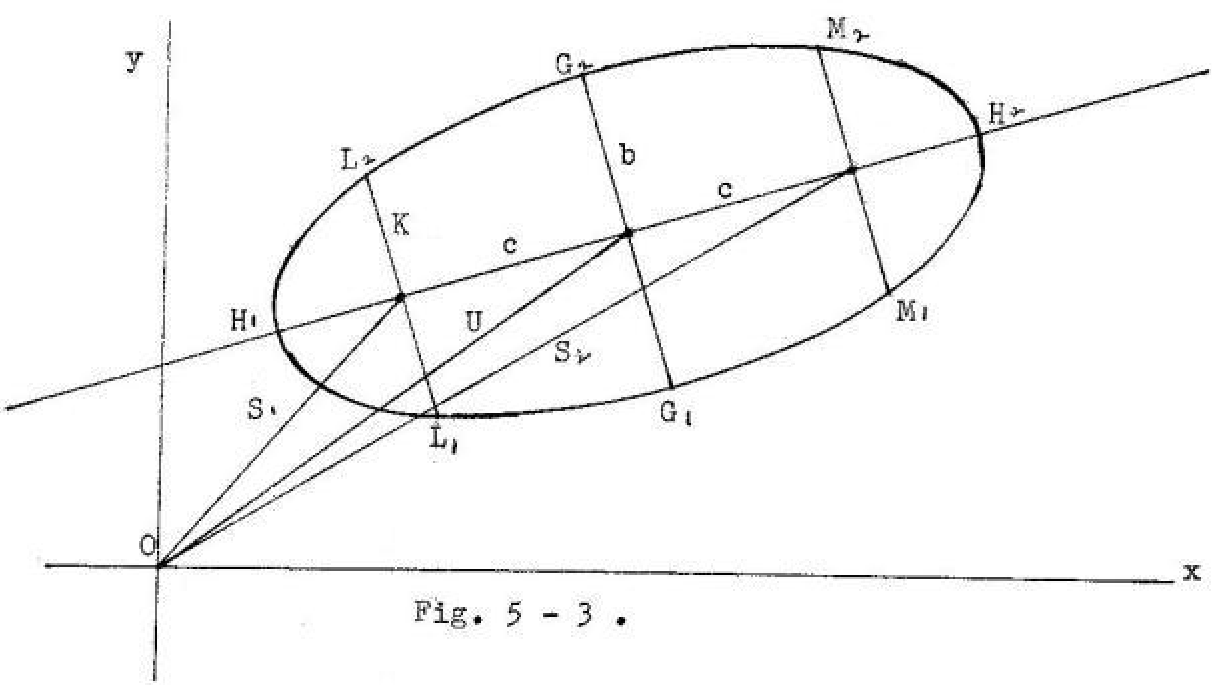

Fig. 5 - 3 .

5 - 5. Equation of the Line Thru the Foci, a Central Diameter

The equation of the line thru the foci may be written :

$$(1) \qquad \breve{e} \cdot r = \breve{e} \cdot U$$

where U is the vector to the center of the conic. e is given by

$$(2) \qquad \breve{e} = - e_2 i + e_1 j = - \sqrt{1 - c}\ i + \sqrt{1 - a}\ j .$$
$$= (- \sqrt{A - C - G_0}\ i + \sqrt{C - A - G_0}\ j)/\sqrt{p} .$$

Our equation (1) may then be written in the form :

$$(3) \qquad - \sqrt{A - C - G_0}\ x + \sqrt{C - A - G_0}\ y =$$
$$- \sqrt{A - C - G_0}\ U_1 + \sqrt{C - A - G_0}\ U_2$$

where

$$(4) \qquad U_1 = (B E - 2 C D) / (B^2 - 4 A C)$$

$$(5) \qquad U_2 = (B D - 2 A E) / (B^2 - 4 A C) .$$

The equation of the minor diameter is the equation of the line thru U perpendicular to the line whose equation is given above. One may solve these two lines with the equation of the given conic to obtain the major and minor vertices of the conic. They should give the same results as those obtained above. For the given conic equation:

$$(6) \qquad 2 x^2 - 4 x y + 5 y^2 = 12 x + 6 y - 42$$

eqvation (3) gives for the equation of the major axis the equation

$$(7) \qquad x = 2 y$$

and the equation of the minor axis is :

$$(8) \qquad 2 x + y = 15$$

If one solves equations (6) and (7) for their points of intersection one obtains:

$$y = 3 \pm \sqrt{15} / 5 = 3.77 \text{ and } 2.23$$

$$x = 2y = 7.54 \text{ and } 4.46$$

which gives the points:

$$(x_1 , y_1) = (7.54, 3.77)$$

$$(x_2 , y_2) = (4.46, 2.23)$$

These answers are identical with those previously given above. In the same way one may solve equations (6) and (8) and determine the vertices of the conic where the minor axis intersects it. It is left as an exercise for the student .

It should be pointed out that when B in the general conic equation

$$A x^2 + B x y + C y^2 = D x + E y + F$$

is postive the equation of the major axis is given by the expression.

$$(9) \qquad \sqrt{A - C - G_0}\, x + \sqrt{C - A - G_0}\, y = \sqrt{A - C - G_0}\, U_1 + \sqrt{C - A - G_0}\, U_2.$$

Since this major axis or central diameter equation is of considerable importance we shall develope another equation which has no reference to the center of the conic. These two equations, when developed, will determine the center of the conic since they each will pass thru the center of the conic. One could, of course, determine the center of the conic from either equation and the equation of the conic by taking one half the sum of the coordinates of the points of intersection of the diameter equation with the equation of the conic.

Any diameter of a conic may be defined as a line which bisects a specified family of parallel chords. When this diameter is the perpendicular bisector of the family of parallel chords it is a central diameter, containing the major or minor axis. See Fig. 5 - 4 for a sketch of the system of parallel chords and the corresponding diameter. The one depicted is not necessarily a central diameter.

We shall look at it from the conventional viewpoint. Some will not want to leave the old mode of thinking and even I write this down with a sentimental nostalgia, knowing full well that it is a lost cause.

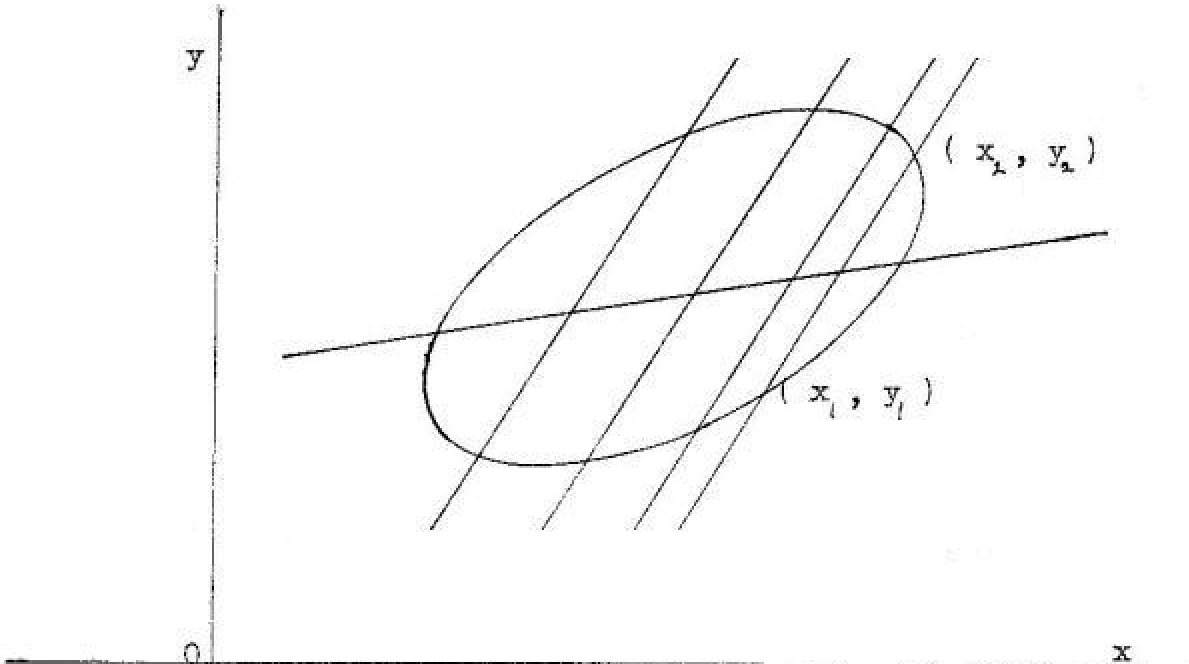

Fig. 5 - 4.

Let

(10) $$A x^2 + B x y + C y^2 = D x + E y + F$$

be the equation of the conic and

(11) $$y = M x + N$$

be the equation of one of a system of parallel chords. Eliminating y from equations (10) and (11) one obtains:

(12) $$x^2 + H x + L = 0.$$

$$H = (B N + 2 C M N - D - E M)/(A + B M + C M^2).$$

$$L = (C N^2 - N E - F)/(A + B M + C M^2).$$

Let the chord, eq. (11), cut the conic, eq. (10), in the points (x_1 , y_1) and (x_2, y_2). The x coordinate of the midpoint of the chord is

(13) $$x = (x_1 + x_2)/2 .$$

By the theory of equations $x_1 + x_2$ is equal to the negative of the coefficient of x in equation (12). Thus from equations (12) and (13) one has:

$$(14) \qquad x_1 + x_2 = 2x = -H.$$

Eliminate N from equations (11) and (14) and one obtains :

$$(15) \qquad y = g x + h$$

$$g = -(B M + 2 A)/(B + 2 C M).$$

$$h = (D + E M)/(B + 2 C M).$$

Since equations (11) and (15) are to be perpendicular we have:

$$(16) \qquad M g = -1.$$

This expands to a quadratic in M whose roots are:

$$(17) \qquad M = (C - A \pm G_0) / B.$$

Equation (15), with the values of M given by equation (17), is the equation of the diameters containing the major and minor axes.

As a test of the theory or rather a sort of confirmation we find the central diameter equations of the following conic:

$$(18) \qquad 2 x^2 - 4 x y + 5 y^2 = 12 x + 6 y - 42$$

$$G_0 = -5$$

$$M = -2$$

$$M = 1/2$$

For M equal to - 2 one obtains

$$g = 1 / 2$$

$$h = 0.$$

With M equal to 1 / 2 one obtains:

$$g = -2$$

$$h = 15$$

With these two sets of values of g and h set into equation (15) one obtains the two following central diameter equations of the conic:

$$y = x / 2$$

$$y = -2x + 15$$

These two equations have already been derived . See equations (7) and (8).

The central diameter equations given by equation (15) are important for one may solve them and the conic obtaining the major and minor vertices of the conic thus localizing the conic in the original coordinate system.

Before we deal with the parabola we shall do a number of illustrative examples pertaining to the ellipse and hyperbola. With the profusion of avenues of approach to a knowledge of the conics the student might conceivably be somewhat baffled as to the optimum road to go. The author of this text book does not know which is the best. We shall explore a few of them and then look back. Hind sights are often satisfying to some of us and we can then say " I told you so " . We begin by finding the primitive state of a number of conics by using the G, G_o , P, p, and T numbers of the conic. The general conic may be represented by:

(19) $$A x^2 + B x y + C y^2 = D x + E y + F$$

For easy reference we write

(20) $$G = (A - C) i + B j$$

(21) $$G_o = \sqrt{(A - C)^2 + B^2}$$

(22) $$P = A + C + G_o$$

(23) $$p = A + C - G_o$$

(24) $$T = 2 (F + (B D E - A E^2 - C D^2) / (B^2 - 4 A C)).$$

We recall also that the primitive state equation is given by:

$$P\,x^2 + p\,y^2 = T \tag{25}$$

Example 1.

Find the primitive state equation for the conic whose equation is

$$2\,x^2 - 4\,x\,y + 5\,y^2 = 12\,x + 6\,y - 42$$

In this case we have the following values for our patameters:

$$G = (2 - 5)\,i - 4\,j = -(3\,i + 4\,j).$$

$$G_0 = -5$$

$$P = 2 + 5 - 5 = 2$$

$$p = 2 + 5 + 5 = 12$$

$$T = 6$$

Our primitive state equation then becomes, after cancelling the common factor 2, the simple equation:

$$x^2 + 6\,y^2 = 3$$

which is an elipse which may be determined in several ways: even by looking at the p values.

Example 2.

Reduce the following equation to the primitive state equation

$$11\,x^2 - 24\,x\,y + 4\,y^2 = -30\,x - 40\,y + 45.$$

$$G = 7\,i - 24\,j$$

$$G_0 = -25$$

$$P = -10$$

$$p = 40, \quad T = -160.$$

The primitive state equation, after cancelling a - 10 from it, becomes

$$x^2 - 4y^2 = 16.$$

This equation is an hyperbola. Its eccentricity is $\sqrt{5} / 2$.

Example 3.

Find the primitive state equation for the following conic equation

$$8x^2 - 12xy + 17y^2 = 20$$

$$G = -9i - 12j$$

$$G_0 = -15$$

$$P = 10$$

$$p = 40$$

$$T = 40$$

$$x^2 + 4y^2 = 4$$

is the primitive state equation. It is an ellipse with eccentricity equal to $\sqrt{3} / 2$.

Example 4.

Find the primitive state of the following conic equation:

$$3x^2 + 12xy - 13y^2 = 135$$

$$G = 16i + 12j$$

$$G_0 = 20$$

$$P = 10$$

$$p = -30$$

$$T = 270$$

$$x^2 - 3y^2 = 27$$

is the primitive state equation. It is an hyperbola.

Example 5

Find the primitive equation of the following conic equation

$$41\ x^2 - 24\ x\ y + 34\ y^2 = 15\ x + 20\ y$$

$G = 7\ i - 24\ j$

$G_o = -\ 25$

$P = 50$

$p = 100$

$T = 25\ /\ 2\ .$

$$x^2 + 2\ y^2 = 1\ /\ 4$$

is the primitive state equation. It is an **ellipse** with $e = \sqrt{2}\ /\ 2$.

Example 6.

Find the primitive equation of the conic given by the equation

$$x^2 - x\ y + y^2 = 6\ x + 12\ y - 66$$

$G = 0\ i - j$

$G_o = -\ 1$

$P = 1$

$p = 3$

$T = \mathbf{36}$

$$x^2 + 3\ y^2 = \mathbf{36}$$

is the primitive state equation. It is an **ellipse** with $e = \sqrt{6}\ /\ 3$.

Example 7.

Determine the primitive state equation for the conic equation:

$$x^2 + 3\,x\,y + 5\,y^2 = 22$$

$$G = -4\,i + 3\,j$$

$$G_o = 5$$

$$P = 11$$

$$p = 1$$

$$T = 44.$$

$$11\,x^2 + y^2 = 44$$

is the primitive state equation. It is an ellipse with $e = \sqrt{10/11}$.

Example 8.

Determine the primitive state equation for the conic equation:

$$0\,x^2 + 4\,x\,y - 3\,y^2 = 8$$

$$G = 3\,i + 4\,j$$

$$G_o = 5$$

$$P = 2$$

$$p = -8$$

$$T = 16.$$

$$x^2 - 4\,y^2 = 8$$

is the primitive state equation. It is an hyperbola with $e = \sqrt{5}\,/\,2$.

Example 9.

Find the primitive state equation for the conic equation:

$$x\,y = 1$$

$$G = 0\,i + j$$

$$G_o = 1$$

$$P = 1$$
$$p = -1$$
$$T = 2.$$
$$x^2 - y^2 = 2$$

is the primitive state equation. It is a hyperbola with $e = \sqrt{2}$.

Example 10.

Determine the primal state equation for the conic equation.

$$x\,y = -1$$

$$G = 0\,i + j$$
$$G_0 = 1$$
$$P = 1$$
$$p = -1$$
$$T = -2.$$
$$x^2 - y^2 = -2$$

is the primitive state equation. It is an hyperbola with $e = \sqrt{2}.$

Example 11.

Determine the primitive state equation for the conic equation:

$$x\,y = -3\,x + 2\,y$$

$$G = 0\,i + j$$
$$G_0 = 1$$
$$P = 1$$
$$p = -1$$
$$T = -12.$$
$$x^2 - y^2 = -12$$

is the primitive state equation. It is an hyperbola with $e = \sqrt{2}$.

It is obvious from the equations (22), (23), and (24) that every conic equation of the form:

$$x\,y = D\,x + E\,y + F$$

has for its primitive the equation:

$$x^2 - y^2 = 2\,(F + D\,E).$$

The resulting equation can be read off practically at sight. Why go to all the trouble of rotating axes in an attempt to simplify the conic when one can see the answer at sight.

5 - 6. Central Diameter Route to a Knowledge of the Conics

By reshaping Equation (3) of the last section or from a strictly Mutation proceedure one may derive the two perpendicular central diameter equations, containing the major and minor axes, as segments of the equations. These equations are:

(1) $$y = L\,x + M$$

(2) $$y = G\,x + H.$$

$$L = (2\,A - N\,B) / (2\,N\,C - B)$$

$$M = (N\,E - D) / (2\,N\,C - B)$$

$$G = (2\,A - n\,B) / (2\,n\,C - B)$$

$$H = (n\,E - D) / (2\,n\,C - B)$$

$$N = (A - C + G_0) / B$$

$$n = (A - C - G_0) / B.$$

If one solves equation (1) with the general conic equation

(3) $$A\,x^2 + B\,x\,y + C\,y^2 = D\,x + E\,y + F$$

one obtains the two points at the ends of the major axis. Let them be (x_1 , y_1), and (x_2 , y_2). Likewise from equations (2) and (3) one obtains the two points at the end of the minor axis. Let them be (x_3 , y_3) and (x_4 , y_4).

The center of the conic is then given by the by the coordinates:

$$(4) \qquad U_1 = (x_1 + x_2) / 2 = (x_3 + x_4) / 2$$

$$(5) \qquad U_2 = (y_1 + y_2) / 2 = (y_3 + y_4) / 2.$$

The semi-major axis is then given by the expression:

$$(6) \qquad 4 a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

The semi-minor axis is given by the expression:

$$(7) \qquad 4 b^2 = (x_3 - x_4)^2 + (y_3 - y_4)^2.$$

The distance of the foci from the center of the conic is given by

$$(8) \qquad c^2 = a^2 \pm b^2$$

The plus being for the hyperbola and the minus for the ellipse. If equation (2) represents the equation of the minor axis we derive from it the following expression for the sensetized unit eccentricity:

$$(9) \qquad e' = (L i - j) / \sqrt{L^2 + 1} .$$

The vectors to the foci are then given by the expression:

$$(10) \qquad S = U \pm c e'$$

where e is given in equation (9) and c and U by (4), (5), and (8 All quantities refer to the original axes. Equation (10) may be written in coordinate form if so desired. The semi-perfolatum is given by the expression:

$$(11) \qquad K = b^2 / a .$$

One can thus calculate all the desired parameters of the conic when the equations of the central diameters are known, and they are known from the equation of the conic.

We carry thru a numerical example to illustrate the theory in the section just developed.

Example 1.

Analyze the following conic from the central diameter view-point.

$$2x^2 - 4xy + 5y^2 = 12x + 6y - 42$$

$$G_0 = -5$$

$$N = (A - C + G_0) / B = 2$$

$$n = (A - C - G_0) / B = -1/2$$

$$L = 1/2$$

$$M = 0$$

$$G = -2$$

$$H = 15$$

The central diameter equations (1) and (2) then become:

$$y = x/2$$

$$y = -2x + 15$$

Solving these two equations with the given conic above one obtains

$$(x_1, y_1) = (4.46, 2.23)$$

$$(x_2, y_2) = (7.54, 3.77)$$

for the vertices of the conic on the focal line and for the vertices on the perpendicular central diameter the following two points:

$$(x_3, y_3) = (5.68, 3.63)$$

$$(x_4, y_4) = (6.32, 2.36)$$

The coordinates of the center of the conic are given by:

$$U_1 = (x_1 + x_2)/2 = (4.46 + 7.54)/2 = 6$$
$$U_1 = (x_3 + x_4)/2 = (5.68 + 6.32)/2 = 6$$
$$U_2 = (y_1 + y_2)/2 = (2.23 + 3.77)/2 = 3$$
$$U_2 = (y_3 + y_4)/2 = (3.64 + 2.36)/2 = 3.$$

The semi-major axis is given by the expression:

$$4\, a^2 = (4.64 - 7.54)^2 + (2.23 - 3.77)^2 = 12$$
$$a = \sqrt{3}.$$

The semi-minor axis is given,by the expression:

$$4\, b^2 = (5.68 - 6.32)^2 + (3.63 - 2.36)^2 = 2$$
$$b = \sqrt{2} / 2.$$

The distance from the center to the focus is given by:

$$c^2 = a^2 - b^2 = 3 - 1/2 = 5/2 = 10/4$$
$$c = \sqrt{10} / 2 .$$

From the minor central diameter equation $y = -2x + 15$ one obtains the unit ~~sensetized~~ sense-tized eccentricity:

$$e' = (2\, i + j)/\sqrt{5}.$$

The focal vectors are then given by the expressions:

$$S = U + c\, e' = (S_1 , S_2) = (7.42, 3.71).$$

$$s = U - c\, e' = (s_1 , s_2) = (4.58, 2.29).$$

See Fig. 5 - 5 for a sketch of this conic.

The drawing below is not drawn to scale . It is only symbolic.

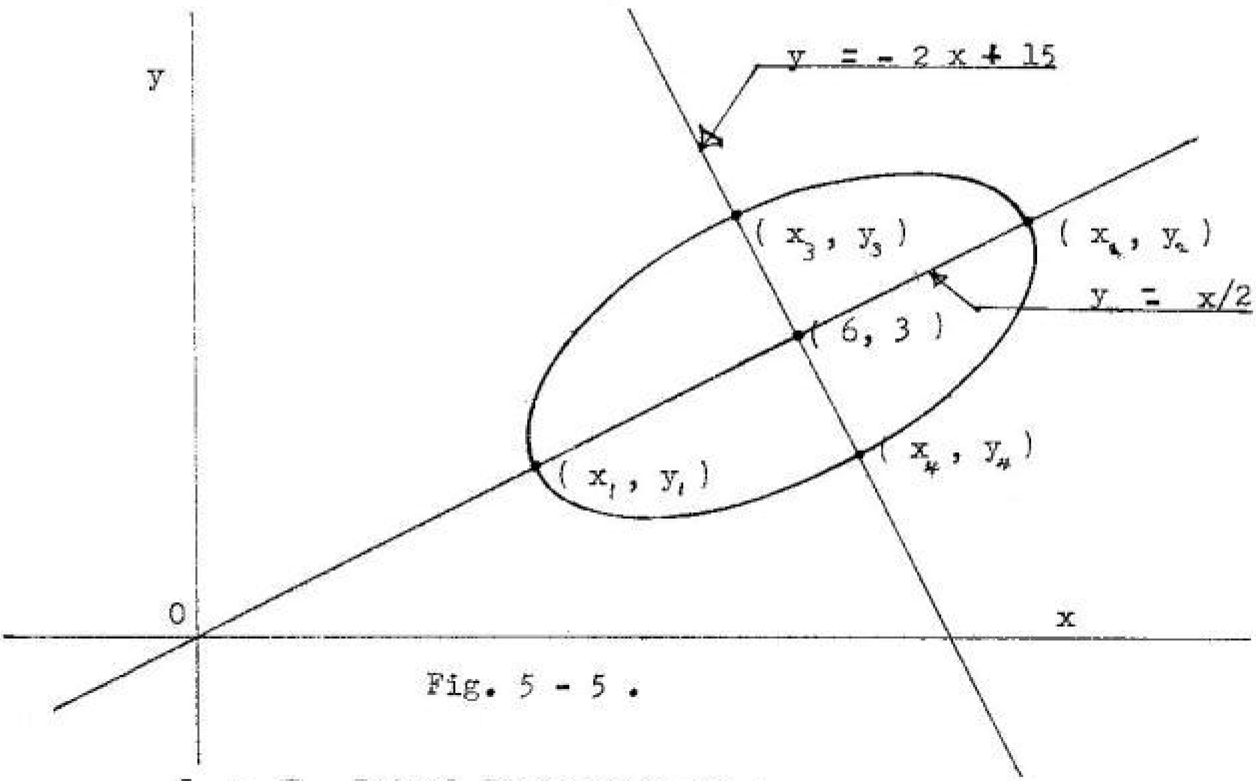

Fig. 5 - 5 .

5 - 7. Primal State Route to a Knowledge of the Conics.

If one multiplies the general equation of the conic (1) below

(1) $$A x^2 + B x y + C y^2 = D x + E y + F$$

by the quantity 2 / P, where P is the larger of the Primal State numbers, one obtains the Primal State Equation for the conic. It is

(2) $$a x^2 + b x y + c y^2 = d x + e y + f.$$

The sensetized eccentricity is given by the expression:

(3) $$e = i\sqrt{1 - a} + j\sqrt{1 - c}$$

from which one obtains the unit sensetized eccentricity e'.

If one squares the equation in (3) the result is

(4) $$\breve{e}^2 = 2 - a - c .$$

From equation (9) in section 5 - 4 one has for a the semi-major axis the expression

(5) $$a = \sqrt{4 f (1 - e^2) + h^2 - (h \cdot \breve{e})^2} \; / \; 2 (1 - e^2).$$

$$h = d i + e j.$$

The e in the expression for h is the coefficient of y in the Primal State equation and should not be confused with the eccentricity e. We may now write down the value of c knowing the value of a and e. It is

(6) $$c = a e.$$

(7) $$b = \sqrt{a^2 - c^2} = a \sqrt{1 - e^2} .$$

One is now in a position to write down the primitive equation if so desired. It is

(8) $$x^2 / a^2 \pm y^2 / b^2 = 1.$$

The center of the conic according to equation (7) in section 5 - 4 is given by the expression:

(9) $$U = (h - h \cdot \breve{e} \breve{e}) / 2 (1 - e^2).$$

One is now in a position to write the expressions for the two foci and the four vertices. They are:

(10) $$S = U + c e'$$

(11) $$s = U - c e'$$

(12) $$H_1 = U + a e'$$

(13) $$H_2 = U - a e'.$$

(14) $G_1 = U + b\,\overset{1}{\breve{e}}$

(15) $G_2 = U - b\,\overset{1}{\breve{e}}$.

If one wants to find the local of the ends of the perfolata he may write them with the expressions:

(16) $L_1 = S + K\,\overset{1}{\breve{e}}$

(17) $L_2 = S - K\,\overset{1}{\breve{e}}$

(18) $M_1 = s + K\,\overset{1}{\breve{e}}$

(19) $M_2 = s - K\,\overset{1}{\breve{e}}$

(2 0) $K = b^2 / a$.

We shall do a numerical example to illustrate the theory in the last section. With a few comments this will bring to a close the analytic theory of the central conics.

Example 1.

Analyze the following conic from the Primal State View-point:

$$2\,x^2 + 4\,x\,y - y^2 = 3\,x + 6\,y - 27/8.$$

$$G = 3\,i + 4\,j$$

$$G_0 = 5$$

$$P = 6$$

$$p = -4$$

$$T = -6$$

The Primitive State Equation is then given by the expression

$$3\,x^2 - 2\,y^2 = -3$$

From this equation one obtains the values of a, b, and c. They are

$$a = \sqrt{3 / 2}$$
$$b = 1$$
$$c = \sqrt{5 / 2} .$$

We now derive the primal State Equation from which we shall derive the sensetized eccentricity and then the above parameters from the Primal State coefficients. Our multiplying factor is:

$$2 / P = 2 / 6 = 1 / 3 .$$

Multiplying the given equation by 1 / 3 we obtain the Primal State Equation. It is

$$2/3\, x^2 + 4/3\, x\, y - 1/3\, y^2 = x + 2\, y - 9/8.$$

$$e = i\sqrt{1 - a} - j\sqrt{1 - c} = (i - 2 j) / \sqrt{3} .$$
$$\breve{e} = (2 i + j) / \sqrt{3} .$$
$$e_o = \sqrt{5 / 3} .$$
$$h = i + 2 j.$$
$$h^2 = 5$$
$$h . \breve{e} = 4 / \sqrt{3}$$
$$1 - e^2 = - 2 / 3.$$

The semi-major axis a is given by the expression:

$$a = \sqrt{4 f (1 - e^2) + h^2 - (h . \breve{e})^2} / 2 (1 - e^2)$$
$$= \sqrt{4 (- 9/8) (- 2/3) + 5 - 16/3} / 2 (- 2/3).$$
$$= \sqrt{3 / 2} .$$ See (9) section 5 - 4 .

$$c = a\, e_o = \sqrt{5 / 2}$$

$$b = \sqrt{c^2 - a^2} = 1.$$

$$U = (h - h . \breve{e}\, \breve{e}) / 2 (1 - e^2).$$

$$U = (i + 2 j - (4/\sqrt{3}) (2 i + j)/\sqrt{3})/ 2 (- 2/3)$$
$$= (5/4) i - (1/2) j .$$

Equations (19) and (20) of section 5 - 4 give the same values for the coordinates of U.

Having calculated from the Primal State Equation the values of a, b, c, e, and U we are now in position to locate the four vertices, the two foci, and the four perfolata points, the points at the ends of the perfola ta. In this case, the conic being a hyperbola, it is obvious that two of the vertices are at infinity or are not in our part of the neighborhood. Perhaps they do not exist. Just to familiarize the student and teacher with the operations of the New Science we shall calculate the two real vertices, the two foci, and the four perfolata points. From equation (12) we have the expression for one vertex:

$$H_1 = U + a e' = (5/4 i - 1/2 j) + \sqrt{3/2} (i - 2 j)/\sqrt{5}.$$

$$= 1.798 i - 1.600 j$$

$$H_2 = U - a e' = (5/4 i - 1/2 j) - \sqrt{3/2} (i - 2 j)/\sqrt{5}.$$

$$= 0.702 i + 0.600 j .$$

$$S = U + c e' = (5/4 i - 1/2 j) + \sqrt{5/2} (i - 2 j)/\sqrt{5}.$$

$$= 1.957 i - 1.914 j$$

$$s = U - c e' = (5/4 i - 1/2 j) - \sqrt{5/2} (i - 2 j)/\sqrt{5}.$$

$$= 0.543 i + 0.914 j$$

$$K = b^2/a = 1 / \sqrt{3/2} = \sqrt{2 / 3}$$

$$L_1 = S + K \overset{\downarrow}{e} = 1.957 i - 1.914 j + \sqrt{2/3} (2i + j)/\sqrt{5} .$$

$$= 2.688 i - 1.547 j$$

$$L_2 = S - K \overset{\downarrow}{e} = 1.957 i - 1.914 j - \sqrt{2/3} (2 i + j)/\sqrt{5} .$$

$$= 1.226 i - 2.281 j .$$

$$M_1 = s + K\overset{\perp}{e} = 0.543\ i + 0.914\ j + \sqrt{2/3}\ (\ 2\ i + j\ )/\sqrt{5}$$

$$= 1.274\ i + 1.281\ j$$

$$M_2 = s - K\overset{\perp}{e} = 0.543\ i + 0.914\ j - \sqrt{2/3}\ (\ 2\ i + j\ )/\sqrt{5}$$

$$= -\ 0.188\ i + 0.547\ j\ .$$

If one determines the central diameter thru the foci by any of the expressions previously developed one arrives at the equation:

$$y = -\ 2\ x + 2.$$

If this is solved withe the given equation of the conic one arrives at the values listed in H_1 and H_2 above. See Fig. 5 - 6 below illustrating the abobe example.

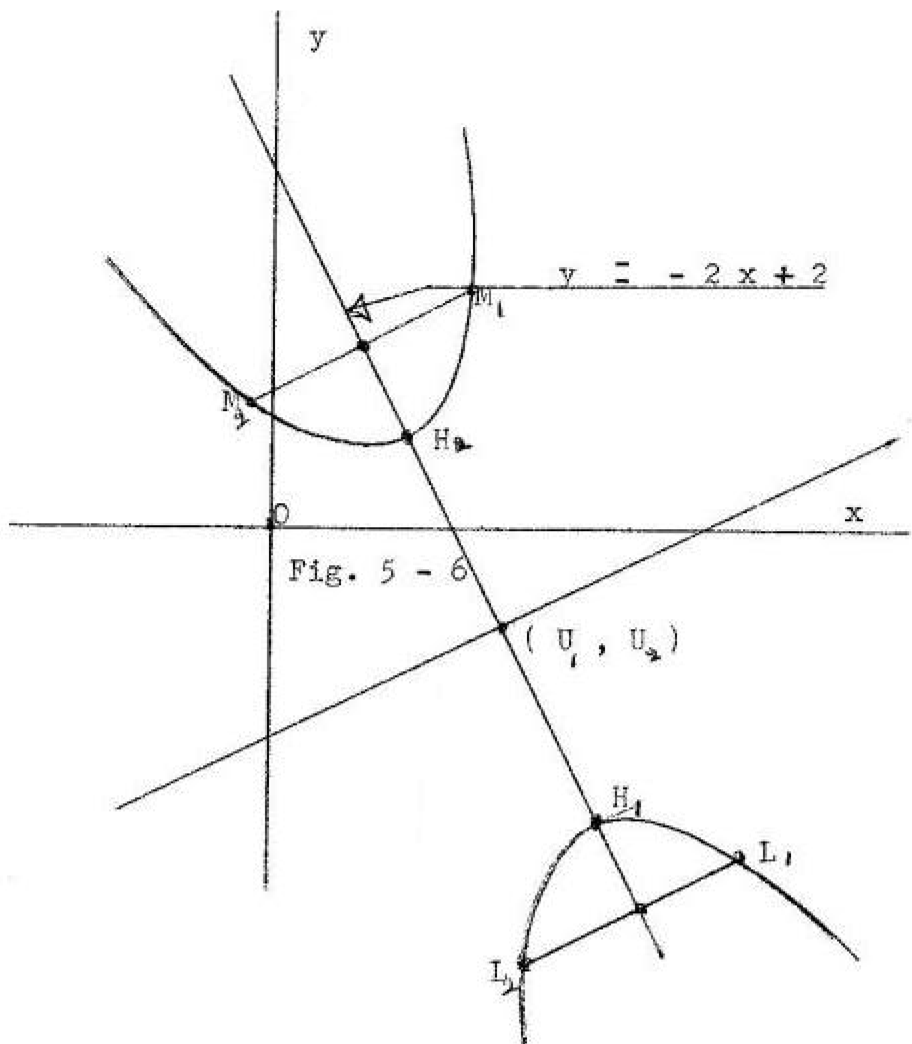

Fig. 5 - 6

It should be pointed out that only the equation given by the expression:

$$P\,x^2 \pm p\,y^2 = T$$

does not apply to the parabola since the parabola is not a central conic. The Central diameter theory applies to the parabola as well as the theory of the Primal State. It is easy to see from the definition of the P and p :

$$P = A + C + G_o$$

$$p = A + C - G_o$$

that p is zero when B = 4 A C for in this case G_o is equal to A + C. T is undefined for the parabola. What then is the form of the above equation when it represents a parabola? We shall see.

5 - 8. Parabola.

If the conic equation:

$$(\ 1\ ) \qquad A\,x^2 + B\,x\,y + C\,y^2 = D\,x + E\,y + F$$

conforms to the following relationship among its coefficients:

$$(\ 2\ ) \qquad B^2 = 4\,A\,C$$

it is said to be a parabola. If equation (2) above is put into equation (2) of section 5 - 3 one sees that the eccentricity e_o is equal to one for the parabola. We write it down below:

$$(\ 3\ ) \qquad e_o = 1.$$

From the expression for the center of a conic:

$$(\ 4\ ) \qquad U = (\ h - h\,.\,\breve{e}\,\breve{e}\ )\ /\ 2\,(\ 1 - e^2\ )$$

it would seem that its center is at an infinite distance away since the factor in the denominator is zero.

We shall not be interested in this great distance at the present but we are interested in the directional part of the expression for U. It has a lot to say. If we multiply this directional part of U by $\breve{e}$ we obtain:

(5) $$\breve{e} \cdot (h - h \cdot \breve{e}\,\breve{e}) = h \cdot \breve{e} - h \cdot \breve{e} \equiv 0$$

keeping in mind that $e^2 = (\breve{e})^2 = 1$. We thus see from (5) that ($h - h \cdot \breve{e}\,\breve{e}$) has the direction of e and, most important, it is the direction of the open end of the parabola.

If I had made a mistake and had put - e for the direction of the open end of the parabola the expression ($h - h \cdot \breve{e}\,\breve{e}$) will correct me and get my mistake right. What " intelligence " in an expression. It will not listen to my mistake.

If one puts the expression $B^2 = 4\,A\,C$ into the expression for G_0 one obtains:

(6) $$G_0 = \sqrt{B^2 + (A - C)^2} = (A + C).$$

(7) $$P = A + C + G_0 = 2 (A + C).$$

(8) $$p = A + C - G_0 = 0.$$

(10) $$2 / P = 1 / (A + C)$$

If we multiply equation (1) by equation (10) we obtain the Primal State Equation:

(11) $$a\,x^2 + b\,x\,y + c\,y = h \cdot r + f$$

(12) $$h = d\,i + e\,j$$

(13) $$r = x\,i + y\,j\,.$$

From the expression $B^2 = 4\,A\,C$ one sees that A and C always has^ve^ the same sign. We shall always write our equations so that they are postive. There is no loss of generality in doing this.

If we replace the Primal State values in the expression for the direction of the open end of the conic (parabola) we obtain the expression:

(14) $$h - h \cdot \breve{e}\,\breve{e} = L (2\,C\,i - B\,j).$$

(15) $$L = (2\,C\,D - E\,B) / 4\,C (A + C)\,.$$

From equation (14) we learn that the open end of the parabola lies in either the first or third quadrants when B is negative and it lies in the first or third quadrant according as L is postive or negative. When B is postive the open end lies in either the second or fourth quadrants. It lies in the fourth or second as L is postive or negative.

5 - 9. Vertex, Focus, Perfolatum

For a complete knowledge of a parabola three essentials suffice: They are the locals of the vertex and the focus and the magnitude of the perfolatum. We look at its focus first. If we multiply (12) of section 5 - 2 by $\breve{e}$ we obtain:

$$(1) \qquad \breve{e} \cdot S = h \cdot \breve{e} / 2$$

Equation (12) Of section 5 - 2 may be written in the form:

$$(2) \qquad e^2 (K - e \cdot S)^2 = (h / 2 - S)^2.$$

Equation (13) of section 5 - 2 may be written in the form:

$$(3) \qquad e^2 (K - e \cdot S)^2 = e^2 (S^2 + f).$$

Comparing the last two equations we obtain:

$$(4) \qquad (h/2 - S)^2 = e^2 (S^2 + f).$$

When e = 1, a parabola, equation (4) reduce to the form:

$$(5) \qquad h \cdot S = h^2 / 4 - f .$$

A solution for the proto-type equations (1) and (5), according to (9), (10), and (11) of section 1 - 5, is :

$$(6) \qquad S = ((h \cdot \breve{e}/2) \breve{h} + (h^2 / 4 - f) e) / (h \cdot e) .$$

$$(7) \qquad H = S - (K/2) e .$$

Here H is the vector from the origin to the vertex.

From equation (13) of section 5 - 2 one easily finds the value of the perfolatum to be:

$$(8) \qquad K = e \cdot S \pm \sqrt{f + S^2}$$

We shall seldom make use of this equation because of the troublesome question of sign in the right side of the equation. Instead, we shall derive it from equation (12) of the same section. This will avoid the troublesome sign question. Equation (12) may be written :

$$(9) \qquad (K - e \cdot S)\, e = h/2 - S .$$

The left side of equation (9) says that the known right side can be factored into e and some other known factor, say M. Then we may write Equation (9) in the form:

$$(10) \qquad (K - e \cdot S)\, e = h/2 - S = M e .$$

Comparing coefficients of e we arrive at the value of K. It is

$$(11) \qquad K = e \cdot S + M .$$

Equations (6), (7), and (11) give all the essentials of a parabola . See the symbolic picture in Fig. 5 - 7 for a representation of the theory of the parabola.

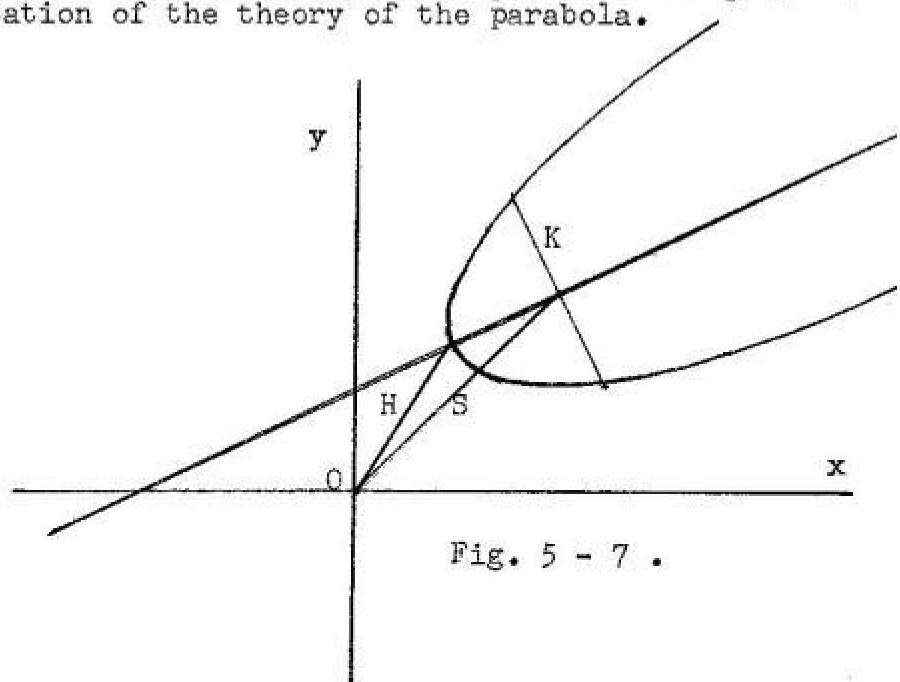

Fig. 5 - 7 .

If in the general conic equation given by

(12) $$A x^2 + B x y + C y^2 = D x + E y + F$$

one takes into account the parabolic relationship:

(13) $$B^2 = 4 A C$$

the eccentricity e given by the expression

(14) $$e = i\sqrt{1 - a} \pm j\sqrt{1 - c}$$

depending on whether B is negative or positive, becomes for caps:

(15) $$e = (2 C i - Bj) / 2\sqrt{C^2 + A C} .$$

Note that e in (15) is unity. Note that it has the same sense, not necessarily the same direction, as the open end of the conic given by equation (14) of section 5 - 8 .

It should be noted that the Primitive State Equation is given by

(16) $$y^2 = 2 K x$$

(17) $$x^2 = 2 K y.$$

These equations are the form taken by the Primitive Equation

(18) $$P x^2 \pm p y^2 = T$$

when the equation is a parabola. We skall do a number of illustrative examples to aid the student and teacher in gaining a feeling for the New Science of Mutation Geometry.

Example 1.

Analyze the parabolic conic whose equation is

$$2 x^2 + 6 x y + 4.5 y^2 = - 7 x - 4 y - 17.$$

From equation (15) we obtain the value of the eccentricity

$$e = (- 3 i + 2 j) / \sqrt{13} .$$

In the given equation B is positive and the L of equation (14) section 5 - 8 is negative. Thus the open end of the parabola points into the second quadrant.

$$G = - 2.5 i + 6 j$$

$$G_0 = 6 . 5$$

$$P = 13$$

$$2/P = 2/13$$

The Primal State Equation is the given by multiplying the given equation by 2/13. We only need the h part of this equation. It is

$$h = - 2 (7 i + 4 j) / 13$$

$$h/2 = - (7 i + 4 j) / 13$$

$$\breve{h} = (4 i - 7 j)/ 6.5$$

$$\breve{e} = - (2 i + 3 j)/\sqrt{13}.$$

From equation (6) of section 5 - 9 one obtains the S. It is

$$S = (- 50.5 i + 25 j)/13$$

$$h/2 - S = (43.5 i - 29 j)/13$$

$$= -(14.5/\sqrt{13}) (- 3 i + 2 j)/\sqrt{13}$$

$$M = - 14.5/\sqrt{13}$$

$$e.S = 201.5/ 13^{3/2} = 15.5/\sqrt{13} .$$

$$K = e.S + M = (15.5 - 14.5)/\sqrt{13} = 1/\sqrt{13} .$$

$$K/2 = 1/ 2\sqrt{13}$$

$$(K/2)\, e = 0.5(-3i + 2j)/13 .$$

$$H = S - (K/2)\, e = (-49i + 24j)/13 .$$

We have now determined the direction of the open end of the parabola, the locals of the focus and vertex, and the magnitude of the perfolatum. The parabola is completely determined.

The central diameter for this parabola is given by the equation

$$2x + 3y = -2.$$

If this equation is solved with the equation of the parabola one obtains the same values for the coordinates of the vertex as those determined above, giving a sort of a confirmation of the first answer.

Example 2.

Analyze the following conic, a parabola:

$$3x^2 + 2\sqrt{3}\, xy + y^2 = 8x - 8\sqrt{3}\, y - 4.$$

$$e = (i - \sqrt{3}\, j)/2$$

$$h/2 = (i - \sqrt{3}\, j)$$

$$S = (5/8)(i - \sqrt{3}\, j)$$

$$h/2 - S = (3/8)(i - \sqrt{3}\, j) = (3/4)(i - \sqrt{3}\, j)/2 .$$

$$M = 3/4$$

$$e \cdot S = 5/4$$

$$K = 3/4 + 5/4 = 2$$

$$(K/2)\, e = (i - \sqrt{3}\, j)/2$$

$$H = S - (K/2)\, e = (i - \sqrt{3}\, j)/8 .$$

We see from the form of e that this conic has its open end in the direction of the fourth quadrant. We shall always identify the direction of e with that of the expression h - h . e e , the direction of the open end of the parabola. From the expressions for S and H it will be seen that they are parallel. This means that the central diameter of this parabola passes thru the origin. One sees from the S and H equations that the focus, being nearer the open end of the parabola, is five times farther away from the origin along the central diameter than the vertex which is nearer the origin on the common central diameter. The central diameter for this equation is:

$$y = -\sqrt{3}\,x .$$

If we solve this equation with the parabolic equation above we obtain the same values for the coordinates of the vertex as those in H. This gives a sort of confirmation of the first result. The central diameter for the parabolic equation

$$A x^2 + B x y + C y^2 = D x + E y + F$$

by any of the expressions already developed for the central diameter, may be written:

$$Y = L x + M$$

$$L = -(B/2C)$$

$$M = (2 C E + B D)/ 4 C (A + C)$$

This is an important equation in that it in conjunction with the equation of the given parabola enables one to determine the vertex immediately. This local of the vertex is an important step in analyzing the conic.

It should be pointed out that the only part of the factor L in the expression for the open end of the conic that determines sign is the expression

$$(2 C D - E B).$$

The part in the denominator is always positive. The sign determiner above is easy to calculate. Be it remembered that the sign of B determines whether the central diameter lies in the direction of the first and third quadrants or the second and fourth quadrants. The quantity (2 C D - E B) makes the final selection. The sense-tized eccentricity e is given this direction.

With example 2 we bring to a close Mutation's view of analytic geometry. In chapter 6 to follow we shall do a number of the common problems of college geometry from the Mutation viewpoint. We shall then use the new science of Mutation Geometry to generalize the more important propositions of college geometry thereby encompassing all that is worth while in this field.