CHAPTER SIX

Mutation Geometry View of College Geometry

6 - 1. The Alpha Postulate in Action

We restate this postulate here: The alpha and omega products are required to be tempo-locally invariant. These products may be shifted mentally from here to yon at any time without changing their value It has an analogue in plane geometry: figures may be moved from one place to another (at any time) without changing their shape.

The alpha products are of the form a . r and the omega products are of the form a . r b . r . We do a typical construction problem:

Problem 1. Construct a rectangle to have a given perimeter and with each of its sides passing thru one of four given points on a plane. See Fig. 6 - 1 below

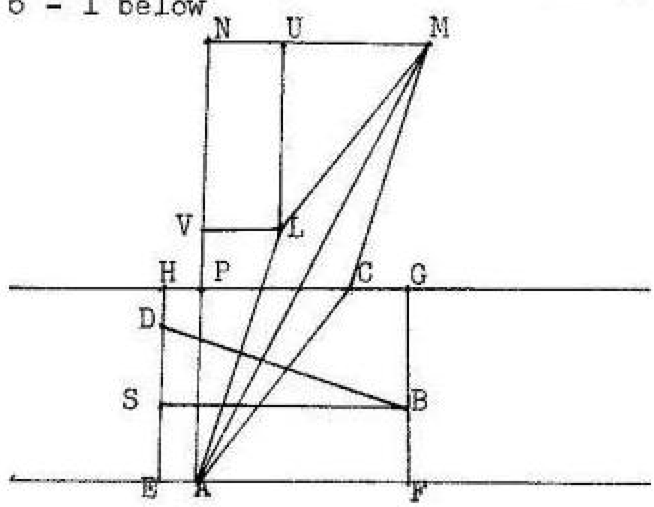

Fig. 6 - 1

Let A, B, C, and D be the four given points. Let EFGH be the required rectangle . Suppose side EF passes thru point A, FG thru B, GH thru C and HE thru D. Designate the vector AC by a and the vector BD by b. Let the unit vector along EH be denoted by r then the unit vector along FE will be denoted by $\check{r}$. Let mn = s denote the semi-perimeter.

From the statement of the problem we may write the following equations:

$$(1) \qquad a \cdot r = AP$$

$$(2) \qquad b \cdot \breve{r} = BS$$

Adding equations (1) and (2) we obtain the following expression:

$$(3) \qquad a \cdot r + b \cdot \breve{r} = s.$$

We may interchange the slurs on the b and r in the second product and obtain the equation:

$$(4) \qquad a \cdot r + \breve{b} \cdot r = s.$$

This act of interchanging slurs is in harmony with the alpha postulate. It preserves local invariance and in this case gives us a common factor r in our single equation. With this change the last equation becomes:

$$(5) \qquad (a + \breve{b}) \cdot r = s.$$

Equation (5) is a composite prototype equation whose solution for r is given in equation (2) in section 1 - 2. We shall instead give a mechanical solution for it since that is the goal of our problem. In Fig. 6 - 1 draw AL perpendicular to and equal to BD. Complete the parallelogram on AL and AC as ALMC. Put a circle on AM as diameter. With A as center and s as radius cut this circle in the two points N and N, the later point not shown in the Fig. Connect M and A to N forming a right angle at N. Draw LU perpendicular to MN. Draw LV perpendicular to AN. Thru A and C draw lines perpendicular to AN. Draw lines thru B and D perpendicular to thes lines forming the rectangle EFGH. Draw BS perpendicular to EH and let GH cut AN in point P. It should be pointed out that AL is $\breve{b}$. We shall now show that the rectangle EFGH has its half perimeter equal to AN. The rectangle satisfies the first requirement that each side should pass thru a given point. Rt. triangles BSD and ALV are congruent since their hypoteni AL and BD were made equal and the sides of the two triangle a re perpendicular. Then

$$AV = BS = EF$$

corresponding parts of congruent triangles being equal.

Rt. triangles ACP and LMU are congruent their hypoteni AC and LM being opposite sides of the parallelogram ALMC and their sides being parallel. We may then write:

$$VN = LU = AP = HE.$$

Adding the last two equations we obtain the result:

$$AN = AV + VN = EF + HE \qquad \text{Q E D. (Nov. 20, 1963).}$$

It should be pointed out that there will be two, one or no solutions according as AN is less than, equal to or greater than AM.

Problem 2. Construct a rectangle each of whose sides shall pass thru a given point and have an area equal to that of a given square. Let the given points be the same as in problem 1. Let h be the side of the given square. With r defined as in problem 1. we may write the following equation:

$$(1) \qquad (a \cdot r)(\breve{b} \cdot r) = h^2$$

Splintering equation (1) with the omega proposition we obtain

$$(2) \qquad a \cdot \breve{b} + \breve{b} \cdot s = 2h^2$$

$$s \wedge a \ r \ , \quad s_0 = a_0 .$$

Equation (2) may be written in the form:

$$(3) \qquad \breve{b} \cdot s' = (2h^2 - a \cdot \breve{b})/a_0 = k$$

Equation (3) is a prototype equation and we may solve it either mechanicly or analytically. We shall do it mechanically. Put a circle on $\breve{b}$ as diameter. With A as center and k as radius cut this circle in the two points N and N', the later not shown in the diagram. AN gives the direction of s. Having found s we bisect the angle between s and a, according to the Mutation Diagram in (2), and this bisector has the direction of r. r is the direction of one side of the required rectangle. The rest of the solution is obvious.

Here the omega proposition played its full part in the solution of this problem.

For some it may be more revealing to write equation (3) in the form:

$$(4) \qquad \breve{b} . s' = 2 h^2 / a_0 - a' . \breve{b} = k .$$

The expression $2 h^2 / a_0$ is a fourth proportion between h, 2 h, and a_0. $a' . \breve{b}$ is the projection of $\breve{b}$ upon a. This k is the difference between two line segments and is easily constructed.

If one clears equation (3) of fractions and transposes one gets equation (2) which is the same as equation (1) by the omega proposition. If one draws the figure for it one simply repeats the proof of the Omega proposition. The elements in the Omega Proposition may be represented by various parameters or combination of parameters, simple or complex. This does not change the essence of the Proposition. To make the drawings one only has to follow the equations. Occasionally, we shall make the drawings for the most interesting problems, For the Generalizations of the Historical Problems we shall make the draw- for all of them. While we are on problems of this type we shall complicate this one somewhat just to practice the student in the New Science.

Problem 3. Construct a rectangle each of whose sides shall pass thru a given point and have an area equal to that of a given sq uare plus the the difference of the areas of the squares constructed on two ajacent sides of the required rectangle. Let A, B, C, and D be the given points. See Fig. 6 - 1 for a picture of the points. Let a, b, and r be defined as in problem 1. We may then write the following representing equation:

$$(1) \qquad (a . r)(\breve{b} . r) = h^2 + (a . r)^2 - (\breve{b} . r)^2$$

Splintering this with the Omega Proposition we obtain:

$$(2) \qquad a . \breve{b} + \breve{b} . s = 2 h^2 + (a^2 + a . p) - (b^2 + \breve{b} . k) .$$

(3)

$s \wedge a$	r		$s_0 = a_0$
$p \wedge a$	r		$p_0 = a_0$
$k \wedge \breve{b}$	r		$k_0 = b_0$

From the mutation diagram in (3) we see that p and s are the same. Replacing p by s in equation (2) we obtain the equation:

$$(4) \qquad a . \breve{b} + \breve{b} . s = 2 h^2 + (a^2 + a . s) - (b^2 + \breve{b} . k) .$$

We may write equation (4) in the form:

$$(5) \qquad c \cdot s' = d$$

$$c = (\breve{b} - a + B) \cdot$$

$$d = (2h^2 - a \cdot \breve{b} + a^2 - b^2)/a_o \cdot$$

$$B_o = b^2 / a_o.$$

B is the co-migrate of $\breve{b}$. From the Mutation Diagram one reads clockwise tha t k goes into s in the same direction that a goes into b . This means that b, the associate of k in its prototype, must migrate into its comigrate B. This means that B makes twice as large an angle with a as it does with $\breve{b}$ and on the same side of a as $\breve{b}$. Thus the magnitude and direction of B is known. Since the magnitude of k is b_o and not a we change it until it is in magnitude equal to a_o. That is why the magnitude of B comes out as b^2 / a_o.

One may now solve equation (5) either mechanically or analytically. To do it mechanically one puts a circle on c as a diameter and with A as a center and a radius d cut the circle in points N and N'. AN and AN' are the directions of s. For the directions of r we bisect the angle between a and AN and a and AN'. These bisectors, according to the Mutation Diagram, are the directions of r and r is the direction of the side of the required rectangle. The rest of the construction is immediate.

One sees from equation (5) that there will be two, one, or no solution to the problem according as d is less than, equal to, or greater than c . The analytical solution to equation (5) is:

$$(6) \qquad s' = (d\,c \pm \breve{c}\sqrt{c^2 - d^2})/c^2.$$

The remaining part of the solution would be as that above. In college geometry the mechanical solution is the most usuable. In a more advanced geometry one may at times need an analytical expression to substitute into another expression. The mechanical solution does not offer this facility.

In this case, as in all cases involving the omega products, the Omega Proposition must and will play its full part. It reduces all the omega products to a sum of alpha prototypes. These, in accordance with the alpha postulate, are assembled or herded into a primordial alpha prototype. It is generally a composite prototype as in equation (5). $c \cdot s'$ is the composite primordial alpha prototype for this problem. Its mechanical and analytical solutions are given above.

All problems in college and projective will be the solution of the alpha prototype. They may be composite but that does not change their essence.

Problem 4. From a point on the circumference of a given circle draw two chords making a given angle with each other and having their sum equal to a given line. See Fig. 6 - 2 below for a picture of the configuration.

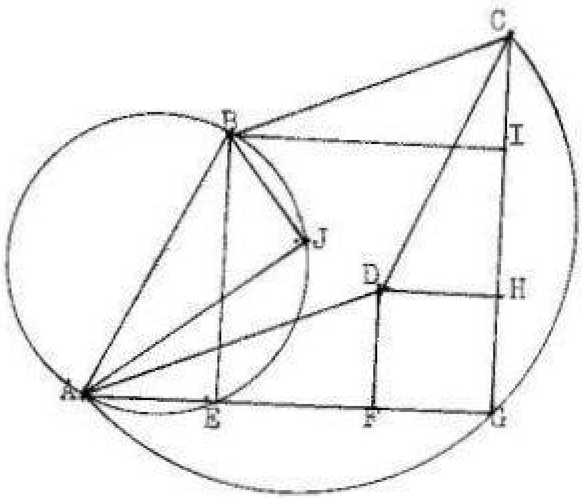

Fig. 6 - 2

Let A be the given point and AB designated by a be the diameter thru the given point A. Let S be the given angle and K the sum of the two chords. Let b and c be the unit directions of the two chords. We may then write:

$$(1) \qquad a \cdot b + a \cdot c = k$$

This may be written in the form

$$(2) \qquad a \cdot b + \widehat{a} \cdot b = k$$

where $\widehat{a}$ is the co-migrate of a . Equation (2) may be written in the primordial prototype form:

$$(3) \qquad d \cdot b = k$$

$$(4) \qquad d = a + \widehat{a} .$$

To construct $\hat{a}$ we draw AD equal to AB and making angle S with AB. AD is then the comigrate $\hat{a}$ in accordance with the definitions of the co-migrates previously given. d, according to equation (4), is the diagonal AC of the parallelogram constructed on AB and AD. The solution of (3) then is the standard mechanical solution of a proto-type. It is: put a circle on AC as diameter and with A as center and k as a radius cut this circle in points G and G′ (G′ not shown in the diagram). Draw AG cutting the given circle in E. Draw the chord AJ in the given circle so that it makes the given angle S with AE. We shall now show that AE and AJ are the two required chords. To do this we shall have to show that the sum of AE and AJ is the line AG.

Proof: Draw BI and DH perpendicular to CG. Draw DF perpendicular to AG. Right triangles ABE and DCH are congruent having their sides parallel and their hypoteni AB and DC as opposite sides of the parallelogram ABCD. From this we have:

(5) $$AE = DH = FG.$$

Right triangles ABJ a nd ADF are congruent having their hypoteni AB and AD equal by construction and angle BAJ is equal to angle DAF both being equal to the given angle S minus the angle DAJ. From this congruency we may write:

(6) $$AJ = AF$$

Adding equations (5) and (6) we obtain:

(7) $$AE + AJ = AF + FG = AG. \quad (Q.E.D.)$$

We have taken pains to go thru this problen in detail. However, I point out tha t we have done nothing but put equation (3) into expanded form. Equation (3) contains every thing contained in these congruencies. It was just a lot of extra labor that we went thru and needs never to be done. When one has reached the primordial prototype its solution conta ins the answer and one must learn to accept it . To persue it futher is a waste of time. One does not need to prove any thing congruent. The solution,of the prototype is the end of the trail . Its solution is the answer.

If the human race had learned or been taught this from its infancy it would not now seem somewhat strange. The mind would need no further satisfaction in a proof. We have now entered the New ERA in our experience. We shall learn to accept the solution of the primordial proto-type as the end of our desires for proof satisfaction.

One could have just as easily solved the problem when each chord went thru a different point on the circumference of the given circle instead of the same point on the circumference. We shall leave this for the student.

While we have Fig. 6 - 2 before us let us solve the problem that will involve the Omega Proposition.

Problem 5. Thru a given point on the circumference of a given circle draw two chords making a given angle with each other and having the product equal to the area of a given square. Let A be the given point and AB the diameter of the given circle. Let h be the side of the given square. See Fig. 6 - 2 for a picture of the configuration. Let b and c, as before, be the unit directions of the chords. We may then write:

$$(1) \qquad a \cdot b \hat{a} \cdot b = h^2$$

Splintering this with the Omega Proposition we obtain:

$$(2) \qquad a \cdot \hat{a} + a \cdot d = 2 h^2$$

$$(3) \qquad d \wedge \hat{a} , b \qquad d_o = a_o$$

One may write equation (2) in the form:

$$(4) \qquad a \cdot d' = (2 h^2 - a \cdot \hat{a}) / a_o = e.$$

The solution of this proto-type equation (4), either mechanically or analytically, gives the value of d'. To find b we see from the MUtation diagram in equation (3) that we must bisect the angle between d and $\hat{a}$. Having found b we draw c making angle S, the given angle, with b. This is the end of the demonstration. We need no proving of congruent triangles. The directions of b and c drawn from point A will strike the given circle in the proper points. Note that in the mechanical solution of this proto-type equation (4) we put a circle on a as diameter (this is the given circle) and with A as a center and e as radius we cut this circle in two points which with A determine the two directions of d. Having found d we then construct the directions of b and c. These determine the two required chords. This is the end of the solution with the observation that there will be two, one or no solution according a s e is less than, equal to or greater than a. One will become used to the new type of solution with practice.

The student should find a new sense of power in the new modes of of operationa in the New Science of Mutation Geometry. He should gain power as he goes. These new modes are designed for the entire field of geometry.

6 - 2. Generalization of the Apollonian Problem.

Historically the Problem of Apollonius was to construct a circle which would be tangent to three distinct circles in the plane. Apollonius of Perga studied the problem and gave a solution. His solution was the end of a series of solutions of simpler cases, interdependent on each other. His solution may be found in most books on college geometry.

It should be pointed out that the word tangent implies both external and internal tangency or what is the same thing it means to cut at zero or 180 degrees. Two circles are tangent externally when their circumferences cut each other at zero degrees. Two circles are tangent internally when their circumferences cut each other at 180 degrees.

It should be further pointed out that the radii of two intersecting circles make with each other the same angle as their circumferences make with each other. This is important. If two circles cut each other at an angle a then the radius of the first circle drawn from the center of the first circle to the point of intersection of the two circles makes angle a with the radius of the second circle drawn from the point of intersection of the two circles to the center of the second circle. See Fig. 6 - 3 for a picture of the configuration.

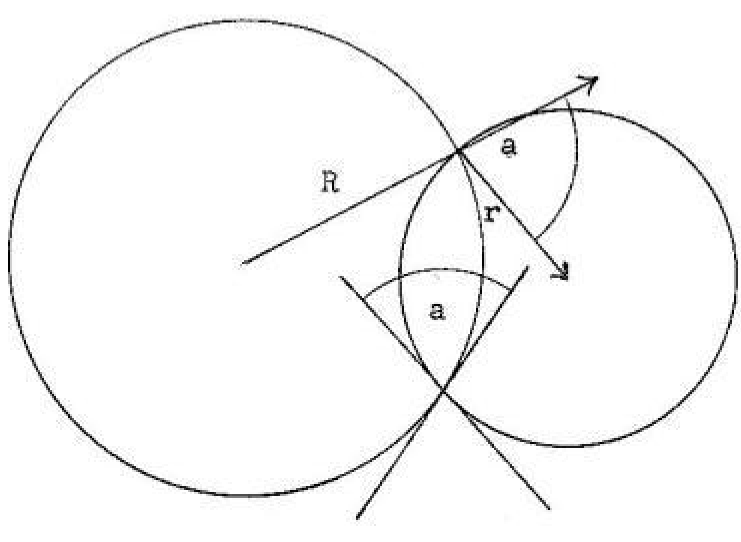

Fig. 6 - 3.

We shall use this fact in our dealings with circles and their properties. It is quite appropriate for our generalizations as we shall see. I would like to think that Apollonius could have generalized his problem but one must remember that he did not have all the accumulated experience that is accessible to modern man. We should be grateful for the solution that he did produce for, only primitive methods were available to him. We shall presently state the generalization.

Generalization of the Problem of Apollonius: Construct a circle which shall cut three given circles at given angles.

Let A, B, and C be the centers of the given circles whose radii are a, b, and c respectively. Let alpha (α), beta (β), and gamma (γ) be the three cutting angles whose cosines are f, g, and h respectively. Let r be the radius of the required circle. Designate AB by e and AC by d. Let M be the center of the required circle and designate AM by U. See Fig. 6 - 4 for a picture of the configuration.

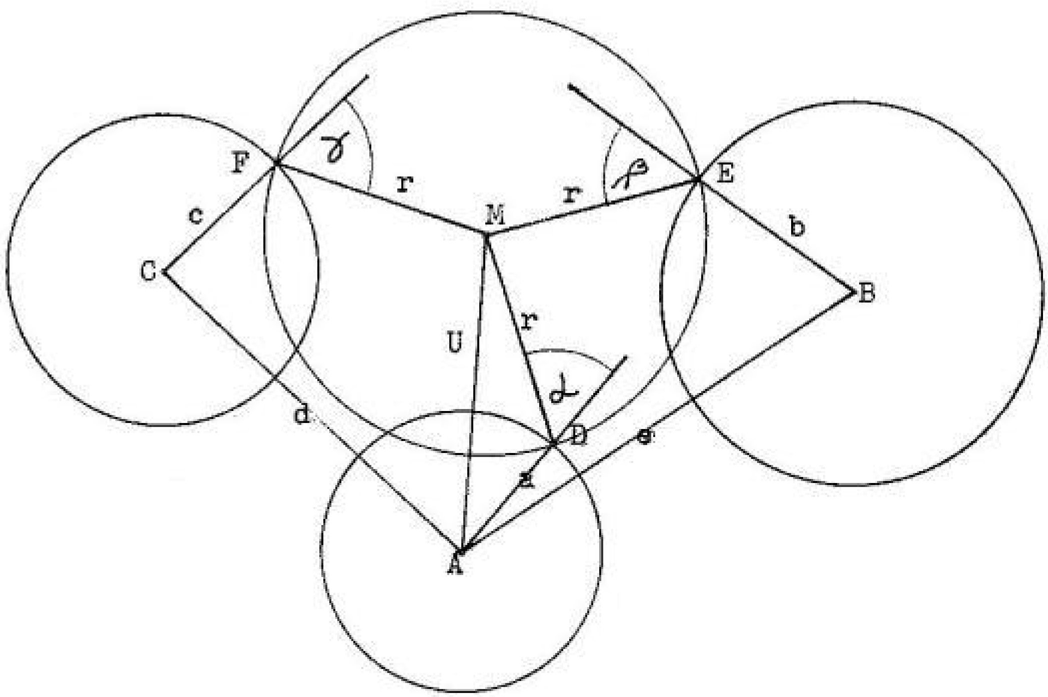

Fig. 6 - 4.

From the configuration we may write the following three equations:

$$(1)\qquad U^2 = a^2 + 2afr_0 + r^2$$

$$(2)\qquad (U - e)^2 = b^2 + 2bgr_0 + r^2$$

$$(3)\qquad (U - d)^2 = c^2 + 2chr_0 + r^2.$$

Subtracting (2) from (1) and (3) from (1) we obtain the two equations:

$$(4) \qquad e \cdot U = p\, r_0 + P^2$$

$$(5) \qquad d \cdot U = n\, r_0 + N^2$$

Solving the prototype equations (4) and (5) for U we obtain

$$(6) \qquad U = G\, r_0 + H$$

$$P^2 = (a^2 + e^2 - b^2) / 2$$

$$p = (a f - b g)$$

$$N^2 = (a^2 + d^2 - c^2) / 2$$

$$n = (a f - c h)$$

$$G = (p \breve{d} - n \breve{e}) / e \cdot \breve{d}$$

$$H = (P^2 \breve{d} - N^2 \breve{e}) / e \cdot \breve{d}$$

Put equation (6) into equation (1) and we obtain the quadratic

$$(7) \qquad r^2 + 2 L\, r_0 - M = 0$$

whose roots are given by the expression

$$(8) \qquad r_0 = - L \pm \sqrt{L^2 + M}$$

$$L = (af - H \cdot G) / (1 - G^2)$$

$$M = (H^2 - a^2) / (1 - G^2).$$

Put the values of r found in equation (8) back into equation (6) and one obtains the two vectors to the center of the required circles:

$$(9) \qquad U = G\, r_0 + H$$

$$(10) \qquad u = G\, r_0 + H.$$

It should be pointed out that when the magnitude of r is known, the magnitude of AM, BM, and CM is known. With A and B as centers and radii AM and BM, one can find the center M of the required circle. This is, perhaps, more convenient than using equation (6) to determine the center M. There will be no solution when M is negative and greater than L^2.

It can also be seen that there is only one circle possible when the circle is to be orthogonal to the three given circles for in this case f, g, and h are all zero which makes G zero and thus L equal to zero. + M takes the value $H^2 - a^2$.

If the required circle is to be tangent to the three given circles f, g, and h will all be either 1 or -1 depending on whether we specify that the required circle is to be tangent externally or internally. For a specified condition there are never more then two circles possible to be constructed to correspond to the specified condition. Eight solutions are possible in the Apollonian problem, but I point out that the word "tangent" to three given circles has several conditions in it. The generalization gives a clear picture of the matter.

There are some people in the geometric world who are still looking for that eighth-degree equation whose roots will give the eight radii of the eight possible circles in the Apollonian problem. A careful look at this Generalization from the Mutation Geometry will resolve this delusion. If A, B and C represent the three given circles and 0 and 180 the cutting angles of the required tangent circle, we have the following possibilities for the Apollonian Problem:

A	B	C
0	0	0
0	0	180
0	180	0
0	180	180
180	0	0
180	0	180
180	180	0
180	180	180

There are 16 (two for each of the assignments) possibilities in all for the Apollonian Problem. The Generalization developed above takes care of all of these cases by assigning the proper values to the f, g, and h which represent the cosines of the cutting angles. There is not even a shadow of an eighth degree equation. In leaving this famous problem we salute Mr. Apollonius of Perga with the Historic Diagram:

The HISTORIC DIAGRAM

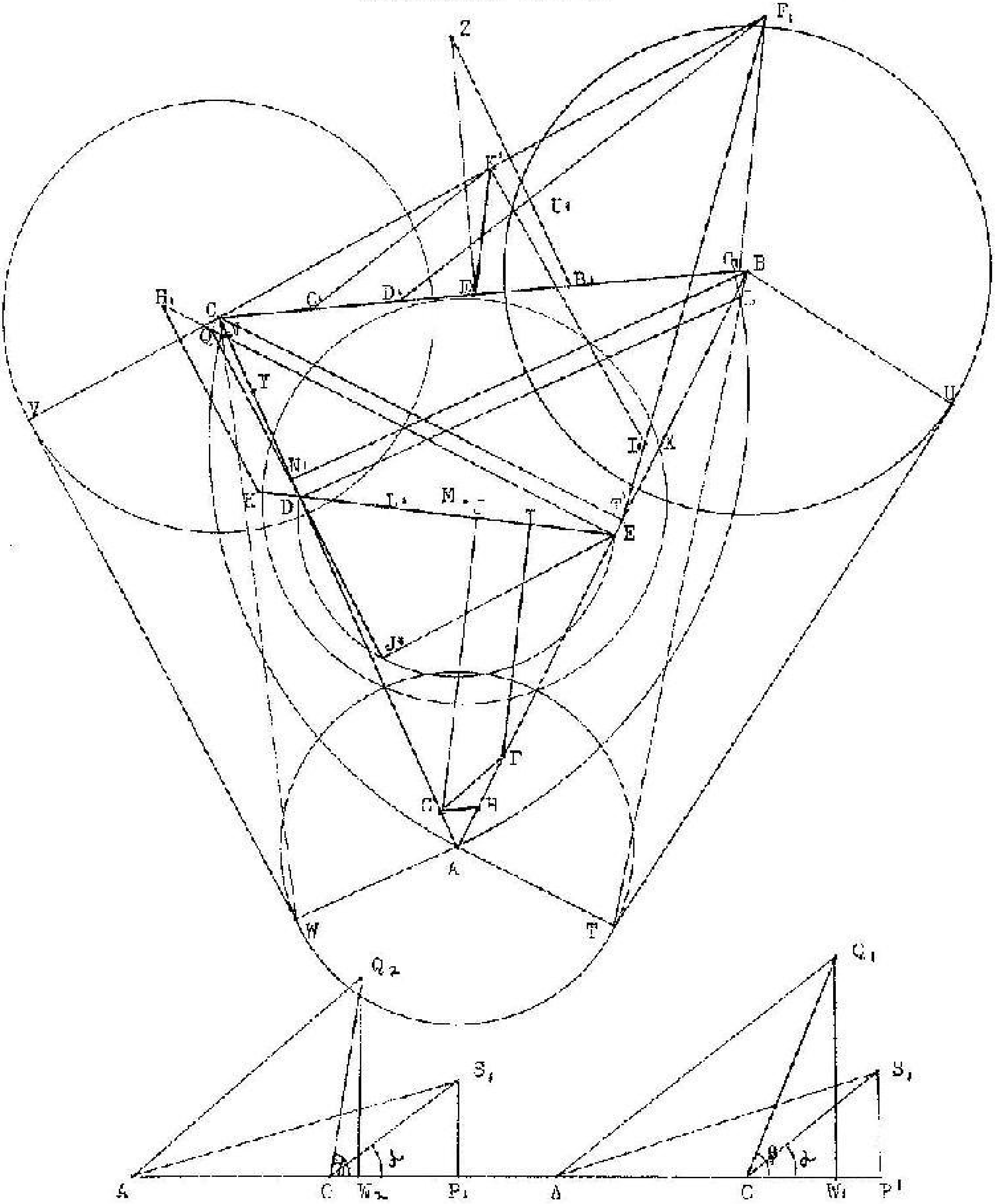

Fig. 6 - 5

6 - 3. Description for the Historic Diagram

The Historic Diagram is based on equation (8). Let A, B, and C be the centers of the given circles whose radii are a, b, and c respectively. The required circle is to cut the given circles A, B, and C at the given angles alpha (α), beta (β), and gamma (γ) respectively.

Draw the altitudes BN' and CT' in triangle ABC. On AB and AC lay off $AX = CT'$ and $AY = BN'$. With X and Y as centers and radii XA and YA draw circles. In circle A draw AT perpendicular to AB and AW perpendicular to AC meeting circle A in T and W. Draw circles on TB and WC as diameters meeting circles B and C in points U and V respectively. With A as a center and radii TU and WV cut the circles on X and Y as centers in points L and N respectively. Draw LD perpendicular to AC and NE perpendicular to AB. Connect E to D. On ED as a diameter draw a circle. With D as a center and a as a radius cut this circle in point J'. Draw EJ'.

Below the main diagram construct the right triangles $S\,O\,P_1$, $Q_1 O W_1$, and $Q_2 O W_2$ where

$$S'O = a$$

$$Q_1 O = b$$

$$Q_2 O = c$$

$$\angle S'OP_1 = \angle \alpha$$

$$\angle Q_1 O W_1 = \angle \beta$$

$$\angle Q_2 O W_2 = \angle \gamma$$

From the last configuration one sees that

$$W_1 P_1 = O P_1 - O W_1 = a f - b g = p \text{ of eq. (6).}$$

$$W_2 P_1 = O P_1 - O W_2 = a f - c h = n \text{ of eq. (6).}$$

On AB lay off $AF = W_2 P_1$ and $AH = W_1 P_1$. Draw HG parallel to BC. Connect F and G. Draw FI and GJ perpendicular to ED. Produce EN to Q making $QE = C\,T'$. Produce EQ to H' making $QH' = JI$. Draw a line thru H' parallel to QD meeting ED in K. Produce KD to L' so that $K L' = O P_1$. On CT' as a diameter draw a circle. With T' as a center and GF as a radius cut this circle in I'. Produce $T_1 I_1$ to F' making $I_1 F_1 = C T_1$.

Draw $I_1 K_1$ perpendicular to CF_1. On (CB) lay off $CC_1 = L_1 D$. Thru F_1 draw a line parallel to $K_1 C_1$ meeting CB in D_1. On CB lay off $CE_1 = EJ_1$. Thru F_1 draw a line parallel to $K_1 E_1$ meeting CB in G_1. On CG_1 as a diameter draw a semi-circle. At E_1 erect a perpendicular to CB meeting this semi-circle in point Z. On CB lay off $E_1 B_1 = C_1 D_1$. On $B_1 Z$ lay off $B_1 U_1 = L_1 D$. ZU_1 is the radius of the required circle. In the triangles below the Historic Diagram produce $B_1 O$ to A making $OA = ZU_1$. With the centers of the three given circles A, B, and C as centers and radii $A S_1$, $A Q_1$, and $A Q_2$ draw three circles all intersecting in the center M of the required circle. With M as a center and radius ZU_1 draw the required circle. The given circles are drawn in green ink and the required circle in red ink.

We now give the reason behind each step of the construction. From the Historic Diagram one sees that $(TU)^2 = (TB)^2 - (BU)^2 = (AB)^2 + (AT)^2 - (BU)^2 = a^2 + e^2 - b^2$. Thus $P^2 = (TU)^2 / 2$. In identically the same way we have the equation $N^2 = (WV)^2 / 2$. Putting the last two values into equation for H in equation (6) we can write H as:

$$H = (TU)^2 \overset{\downarrow}{d} / 2 (e \cdot \overset{\downarrow}{d}) - (WV)^2 \overset{\downarrow}{e} / 2 (e' \cdot \breve{d}) .$$

$$A D = (TU)^2 / 2 (e \cdot \overset{\downarrow}{d})$$

$$A E = (WV)^2 / 2 (e' \cdot \breve{d})$$

We thus may write for H the expression:

$$H = (A D) \overset{\downarrow}{d} - (A E) \overset{\downarrow}{e} = (\overset{\smile}{E D}).$$

It should be pointed out that $e \cdot \overset{\downarrow}{d}$ and $e' \cdot \breve{d}$ are the altitudes of triangle ABC drawn respectively from Band C. From the manner of constructing GF in the Diagram we see that G in equation (6) is given by

$$G = (\overset{\smile}{GF}) / (e' \cdot \breve{d}) = \overset{\smile}{GF}/CT' = \overset{\smile}{GF}/EQ .$$

$$H.G = +(ED) (JI)/ EQ = +(ED) (QH_1)/ EQ = +KD .$$

$$L_1 K = CP_1 = a f$$

$$(a f - H . G) = L_1 K - K D = L_1 D .$$

We must now construct the denominator $(1 - G^2)$ of L in equation (8).

From the value of G given above we have:

$$G = (\overset{\smile}{GF}/CT_1) = (T_1 I_1 / CT_1).$$

$$1 - G^2 = ((CT_1)^2 - (T_1 I_1)^2)/(CT_1)^2 = (CI_1)^2/(CT_1)^2.$$

$$= (CI_1)^2 / (F_1 I_1)^2.$$

In the similar right triangles $C I_1 K_1$ and $F_1 I_1 K_1$ we have the relations

$$(CI_1)^2 = (CK_1)(CF_1)$$

$$(F_1 I_1)^2 = (F_1 K_1)(CF_1).$$

Taking the ratio of the last two equations we obtain:

$$1 - G^2 = (CI_1)^2/(F_1 I_1)^2 = (CK_1)/(F_1 K_1).$$

Our expression for L in equation (8) may now be written:

$$L = (L_1 D)(F_1 K_1)/(CK_1) = (C_1 D_1).$$

We now look for the expression of M in equation (8). The value of $H^2 - a^2$ is given by:

$$H^2 - a^2 = (ED)^2 - (DJ_1)^2 = (EJ_1)^2 = (CE_1)^2.$$

$$M = (H^2 - a^2)/(1 - G^2) = (CE_1)(CE_1)(F_1 K_1)/(CK_1) = (CE_1)(E_1 G_1)$$

$$= (E_1 Z)^2.$$

$$\sqrt{L^2 + M} = \sqrt{(C_1 D_1)^2 + (E_1 Z)^2} = \sqrt{(E_1 B_1)^2 + (E_1 Z)^2} = B_1 Z.$$

$$r_0 = E_1 Z - C_1 D_1 = B_1 Z - E_1 B_1 = B_1 Z - B_1 U_1 = Z U_1.$$

Having found the expression for the radius r of the required circle we turn to the triangles below the main drawing of the Historic Diagram.

The prescription for finding the center of the required circle, after it's has been found, has already been given. It is to produce P_1 O to A so that OA will equal the radius ZU_1 of the required circle. Then with A, B, and C, the centers of the given circles, as centers and radii AQ_1, AQ_2, and AQ_3 as radii draw circles meeting in the common point M, the center of the required circle. With M as a cente and ZU_1 as a radius draw the required circle.

It will be observed from equation (8) that there are at most two circles for any set of chosen angles alpha, beta, and gamma. The conditions for two, one, and no circles are inherent in equation (8) and need not be discussed here.

No doubt, in time, students of geometry will simplify the drawing for the Historic Diagram. Other Mutation expressions may be developed for the Historic Diagram, and they may or may not be simpler than the one here first developed. We shall not spend any time in either simplifying what we have or in writing out other developements. We shall leave this to those who are primarily interested in the field of geometry. The geometric world has had, perhaps, four thousand years in which to write this generalization.

We shall do a couple more generalizations in the field of college geometry which should encompass about all that is worth while in this realm. We shall then take a quick look at the field commonly known as projective geometry. We shall look at it Mutationwise.

6 - 4. Generalization of the Simson Line Theorem from the Mutation Geometry Standpoint.

In college geometry it is proven that the feet of the perpendiculars to the sides of a triangle from a point on the circumference of the circumcircle lie in a straight line. We generalize this to:

The points of intersection with the sides of a triangle of any three lines drawn from a point on the circumference of the the circumcircle and making equal angles with the sides of the triangle lie in a straight line. When the angles are all rt. angles the theorem becomes the Simson line theorem.

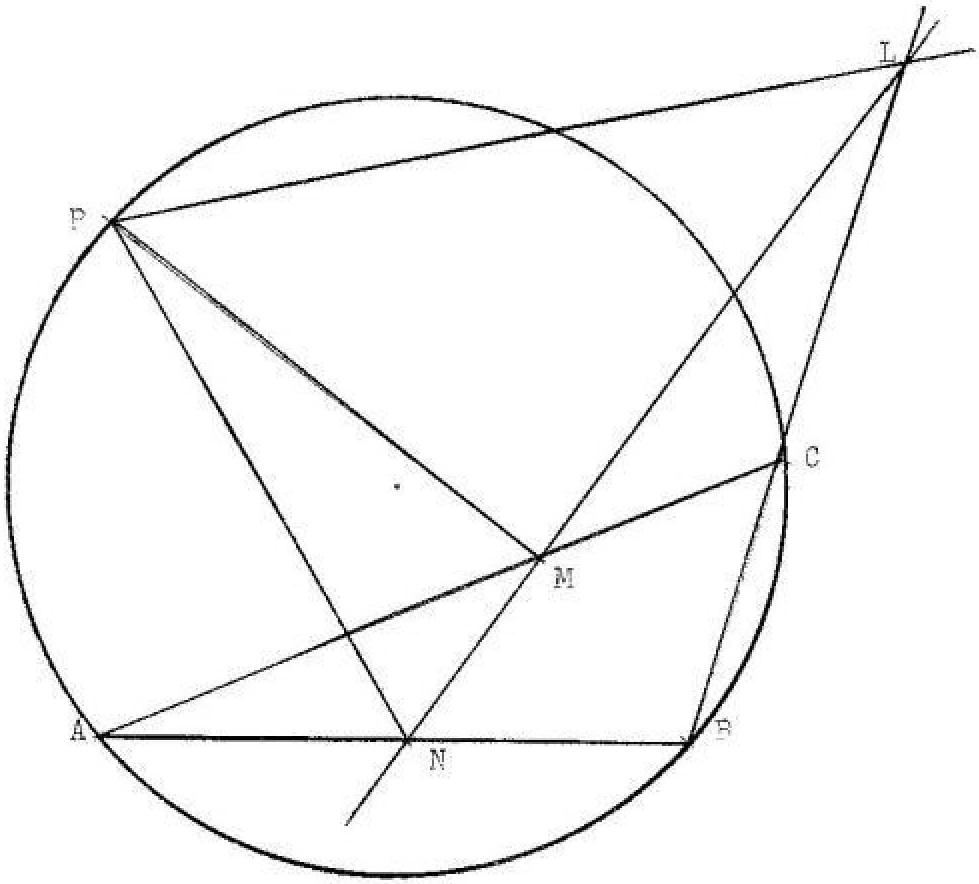

Fig. 6 - 6.

Let ABC be the given triangle and P the given point on the circumference of the circumcircle. See Fig. 6 - 6. We shall prove that the three lines P-LMN making equal angles with the sides BC, CA, and AB have the points L, M, and N in a straight line. Designate the lines PA, PB, PC, PL, PM, and PN by a, b, c, l, m, and n respectively. Let K be the sine of the angle which PL makes with the side BC then from the triangle PBC we may write; considering its area:

$$(1) \qquad l_o = (c.\breve{b})/K (b - c)_o .$$

We may also write the following identity equation:

$$(2) \qquad (b - c)^{\smile\prime}.l^{\prime} = K$$

Solving (2) for l' and the multiplying the result by the value of l_o in equation (1) we obtain the value of l in the form:

$$(3) \quad (c . \breve{b})^{-1} (b - c)^{2} l = \breve{b} - \breve{c} \pm h (b - c).$$

Here h is the cotan of the angle whose sine is K. Using the other two sides of the triangle we may write two more equations similar to equation (3). They are:

$$(4) \quad (c . \breve{a})^{-1} (a - c)^{2} n = \breve{a} - \breve{c} \pm h (a - c).$$

$$(5) \quad (b . \breve{a})^{-1} (a - b)^{2} m = \breve{a} - \breve{b} \pm h (a - b).$$

If equation (4) is subtracted from the sum of equations (3) and (5) one obtains:

$$(6) \quad (c.\breve{b})^{-1}(b-c)^{2} l - (c.\breve{a})^{-1}(a-c)^{2} n + (b.\breve{a})^{-1}(a-b)^{2} m = 0.$$

We now determine the relationship between the coefficients of l, m, and n in equation (6). They are constants for any fixed position of the given triangle ABC and point P. One way would be to expand them and take into account that a, b, and c are chords of a circle. A slightly easier way is the following. One notes from equation (6) that all signs of the size of the equal angles have vanished and we may make them any equal angles, including 90 degrees, in which case L, M, and N are collinear; Simsons Theorem, and l, m, and n are termion-collinear and the sum of their coefficients in (6) is zero and since they are constants their sum will always be zero then l, m, and n in (6) are termino-collinear for any equal angles for the sum of their coefficients is zero and this is the Generalization.

Below we give a visual demonstration of this Generalization. In the cyclic quadrilateral PLCM we have

$$(7) \quad \angle PLM = \angle PCM = \angle PCA = \angle PBA = \angle PBN.$$

From the cyclic quadrilateral PLBN we have the equation

$$(8) \quad \angle PLN = \angle PBN$$

Comparing equation (7) and (8) we obtain the relation

$$(9) \quad \angle PLM = \angle PLN$$

Thus points L, M, and N are collinear.

We shall do one more generalization in the field of college geometry, that of the Brocard Theorem, and with this we shall bring to a close our section on college geometry.

6 - 5. Brocard Generalization

In college geometry it is shown that the three circles on chords AD, AB, and BC, tangent respectively to the sides AB, BC, and AC, pass thru a common point. Tangency means cutting at zero or one hundred and eighty degrees. For the Brocard Theorem the sum of these cutting angles is:

$$(1) \qquad 0 + \theta + 0 = 0$$

In the Generalization we shall show that the three angles M, N, and P, made with the sides AB, BC, and AC by the circles, having a common point, on chords AC, AB, and BC respectively, are related by the equation:

$$(2) \qquad M + N + P = 0$$

where those angles falling outside the triangle are counted as negative. (See Fig. 6 - 7) :

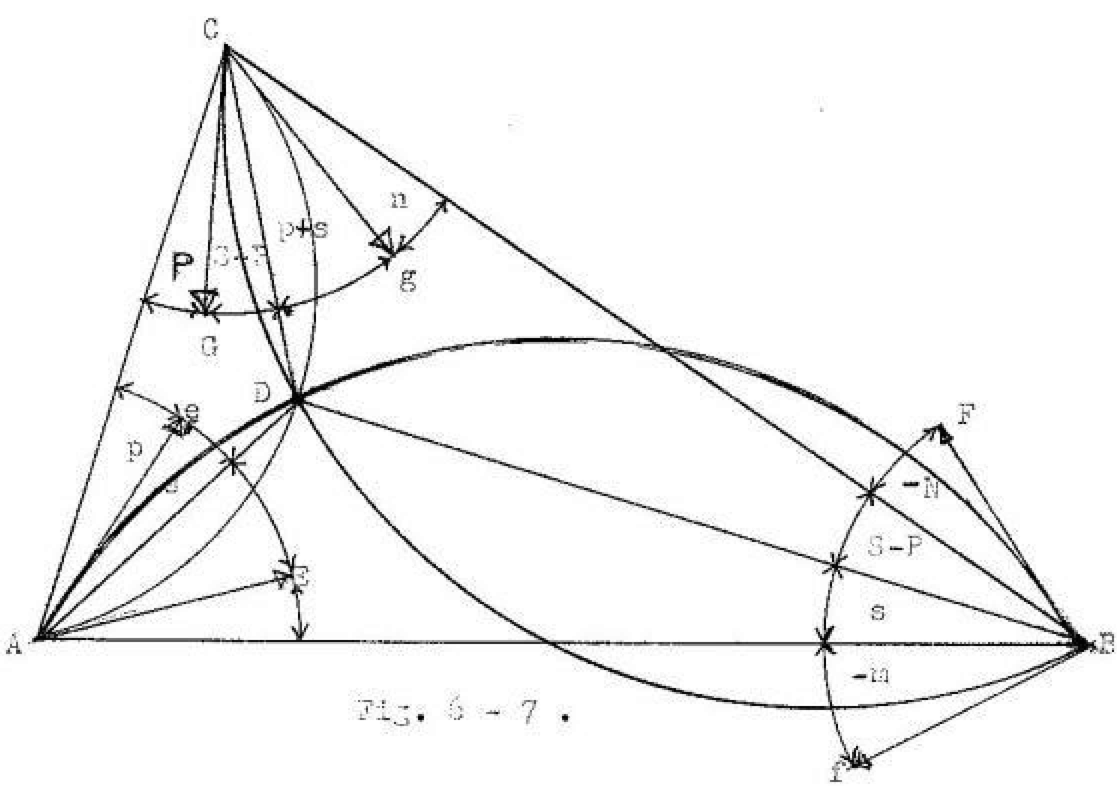

Fig. 6 - 7 .

Let D be the common point of the three circles. At A draw the tangent AE to the circle ADC meeting the side AB in the angle BAE equal to angle M. At B draw the tangent BF to the circle ADB meeting the side BC in the angle CBF equal to angle N. At C draw the tangent CG meeting the side CA in the angle ACG equal to angle P. For ease in writing our equations we shall set angle DAE equal to S. We shall now prove relation (2).

Proof:

Consider arc ADC.

(3) $$\angle DCA = \angle DAE$$

since AE is tangent to the arc at A. Then

(4) $$\angle DCG + \angle P = \angle S \quad \text{or}$$

(5) $$\angle DCG = \angle S - \angle P = \angle (S - P)$$

and is so marked in the figure. Now consider arc BDC.

(6) $$\angle DCG = \angle DBC$$

since CG is tangent to the arc at C. Then from (5) and (6)

(7) $$\angle DBC = \angle (S - P)$$

and is so marked in the figure. Lastly consider arc ADB.

(8) $$\angle DAB = \angle DBF$$

since BF is tangent to the arc at B. Then

(9) $$\angle S + \angle M = \angle (S - P) - \angle N \quad \text{or}$$

(10) $$\angle (M + N + P) = 0 .$$

Equation (10) is the Generalization of the Brocardian Theorem. When

(11) $$\angle M = \angle N = \angle P = 0$$

we get the direct group of ajoint circles of the Brocard Theorem and point D coincides with the direct Brocard point of the triangle. If one had traversed the triangle ABC in the reverse direction one could obtain three circles, passing thru a common point D' and cutting the sides AC, BC, and AB, in the angles P', N', and M', such that

$$(12) \qquad \angle(M' + N' + P') = 0$$

When

$$(13) \qquad \angle M' = \angle N' = \angle P' = 0$$

we obtain the indirect group of circles of the Brocard Theorem and point D' coincides with the indirect Brocard point of the triangle ABC. The indirect Brocard point and circles are not drawn in the Fig. 6-7. In the indirect case the circle on the chord AB cuts side AC at angle P' and the circle on chord AC cuts side BC at angle N' and the circle on chord BC cuts side AB at angle M', all the circles passing thru the common point D'. Let AE' be the tangent at A to the circle on chord AB making angle CAE' equal to angle P'. Let CG' be the tangent at C to the circle on chord AC and making angle BCG' equal to angle N'. Let BF' be the tangent at B to the circle on chord BC making angle ABF' equal to angle M'.

Every consequent deduction that comes from the Brocard Theorem has its counterpart in this Generalization and even more. We give an illustrative example. In college geometry it is proven, as a corallary, that the direct and indirect Brocard points are isogonal congugate points of the triangle.

Let M, N, and P be the direct cutting angles corresponding to point D and AE, BF, and CG the corresponding tangents as shown in the diagram above. Now consider a second point D', not shown in the diagram, whose indirect cutting angles are P', N', and M', the corresponding tangents being AE', CG', and BF'. Set angle DAE' equal to S'. Angles P', N', and M' are, as yet, wholy independent of the angles M, N, and P.

Perhaps it is not generally observed that there is a relation, in disguise, between the direct and indirect cutting angles of the circles for the isogonal congugate points of the Brocard Theorem namely that each direct cutting angle is equal to the corresponding indirect cutting angle each being zero. A natural Generalization for the Brocard isogonal congugate point theorem is to find the relation between the direct cutting M, N, and P and the indirect cutting angles P', N', and M' when D and D' are isogonal congugate points of the six tangents, two at each vertex, of triangle ABC since these coincide with the sides of the triangle ABC when D and D' coincide with the isogonal congugate Brocard points. We shall see. We may write the following equations:

$$(14) \qquad \angle DCG = \angle(S - P)$$

$$(15) \qquad \angle D'CG' = \angle(S' + P')$$

If D and D' are isogonals of the tangents CG and CG' we must have

$$(16) \qquad \angle(S' + P') = \angle(S - P)$$

Again

(17) $$\angle D'BF' = \angle DBA + \angle ABF' = \angle (S' - M')$$

and

(18) $$\angle DBF = \angle DBC + \angle CBF = \angle (S - P) - \angle N .$$

If D and D' are isogonals of the tangents BF and BF' we must have:

(19) $$\angle (S' - M') = \angle (S - P) - \angle N .$$

Adding equations (16) and (19) one obtains

(20) $$\angle (2 S' + P' - M') = \angle (2 S - 2 P - N)$$

If D and D' are isogonals of the tangents AE and AE' we must have:

(21) $$\angle S' = \angle S$$

Taking account of equation (21) equation (20) becomes

(22) $$\angle (P' - M') = - \angle (2 P + N).$$

Adding ($N' + 2 M'$) to both sides of equation (22) and taking into account equation (12) one obtains:

(23) $$\angle (N' + 2 M') = \angle (N + 2 P).$$

Put equation (21) into (16) and we obtain:

(24) $$\angle P' = - \angle P.$$

Put equation (21) into (19) and take account of (2) and we obtain:

(25) $$\angle M' = - \angle M$$

Put equation (25) into (23) and take account of (2) and we obtain:

(26) $$\angle N' = - \angle N .$$

Equations (24), (25), and (26) give us the desired relations. The striking beauty of their symmetry will not be lost on the reader. Now when the direct cutting angles M, N, and P are zero and the direct tangents collapse onto the sides of the given triangle ABC and point D coincides with the direct Brocard point of triangle ABC it is seen from equations (24), (25), and (26) that the indirect cutting angles M', N', and P' are zero and the indirect tangents collapse onto the sides of the given triangle ABC and point D' coincides with the indirect Brocard point thus showing that this particular corollary of the Brocard theorem which we are illustrating is a very special case of a far greater generalization. Other corollaries of the Brocard theorem in a similar way may be shown to be particular cases of a more general viewpoint. All these deductions follow from equation (2) which is the Generalization of the Brocard theorem.

While this textbook on Mutation Geometry is largely devoted to the analytic theory of the conics we have done a number of generalizations in the field of college geometry just to show its versa tility. Before we bring this text to a close we shall take a look at the field of projective geometry from the Mutation Geometry viewpoint.

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