CHAPTER SEVEN

Mutation Geometry View of the Field of Projective Geometry

7 - 1. Comments re-Projective Geometry.

One could state and prove here the principal theorems of projective geometry such as those of Desargues, Pascal, Brianchon, and others by means of which the properties of geometric figures are derived. These properties furnish the means by which various geometric constructions may be made. This, however, would defeat our purpose for if we proved these propositions and used their properties to do geometric constructions we would be doing things in a conventional manner. We should then be under the bondage of the necessity for logical order and sequence. In Mutation Geometry there is to be no such thing as a necessity for logical order and sequence. We shall neither prove the propositions nor shall we use their properties to do desired geometric constructions. We shall show how to do them Mutationwise.

There is no implication in the above statements that the Pascal, Brianchon, and other propositions are not important nor their properties useful. I thoroughly appreciated their beauty and elegance when I was doing a masters degree in mathematics at Indiana University. Be it further pointed out that I was doing graduate work in mathematics before I was able to do some of the constructions demanded in projective geometry. That is a long build up to accomplish a simple construction like finding the points of intersection of a line with a conic given by five points when the conic is not drawn.

We shall first build the equation of a conic thru five points then reshape it so that it will represent the equation of a conic thru four points and tangent to a line thru one of the points or a conic thru three points and tangent to lines thru two of the points. It will then subsequently be shown how to deal with a conic tangent to five given lines or tangent to four line and passing thru the point of contact of one of the given lines or a conic tangent to three given lines and passing thru the points of contact on two of the given lines. It may all be reduced to the equation of a conic thru five given points. We shall then lay the major emphasis on the equation of a conic thru five given points.

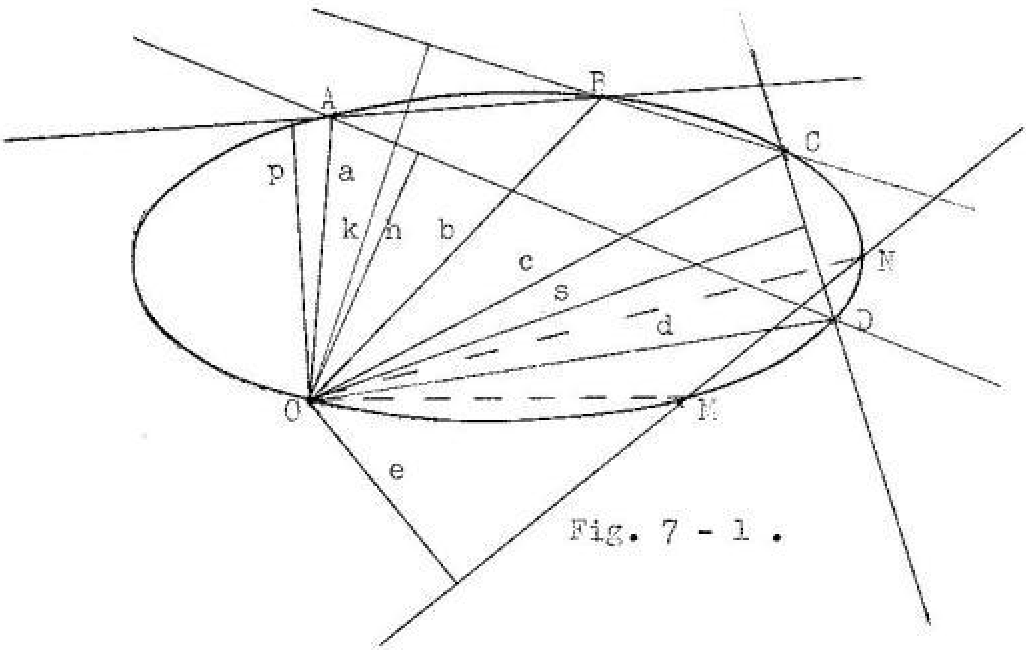

Fig. 7 - 1 .

7 - 2. Equation of a Conic Thru Five Given Points

Let O, A, B, C, and D be the five given points (see Fig. 7 - 1). Let a, b, c, and d designate the four vectors OA, OB, OC, and OD respectively.

Let p, h, s, and k be the vector perpendiculars from O to AB, AD, CD, and CB respectively. Then

$$(1) \qquad p' \cdot r - p_0 = 0$$

$$(2) \qquad s' \cdot r - s_0 = 0$$

are the equations of the lines AB and CD respectively. Their product:

$$(3) \qquad p' \cdot r s' \cdot r - (p_0 s' + s_0 p') \cdot r + p_0 s_0 = 0$$

is the equation of a conic thru the four points A, B, C, and D. In the same way

$$(4) \qquad h' \cdot r - h_0 = 0$$

$$(5) \qquad k' \cdot r - k_0 = 0$$

are the equations of the lines AD and CB respectively. Their product:

$$(6) \qquad h' \cdot r k' \cdot r - (h_0 k' + k_0 h') \cdot r + h_0 k_0 = 0$$

is the equation of a conic thru the four points A, B, C, and D.

Any linear combination of the two conics (3) and (6) will pass thru the four points A, B, C, and D. Forming a linear combination of (3) and (6) we have:

$$(7) \qquad n (r \cdot p' s' \cdot r - (p_0 s' + s_0 p') \cdot r + p_0 s_0) = (r \cdot h' k' \cdot r - (h_0 k' + k_0 h') \cdot r + h_0 k_0).$$

We wish (7) to pass thru point O (r = O). Putting r = O into (7) and we obtain:

$$(8) \qquad n = h_0 k_0 / p_0 s_0 .$$

Equation (7) now takes the form:

$$(9) \qquad r \cdot H \cdot r = G \cdot r$$

$$(10) \qquad H = (n p' s' - h' k')$$

$$(11) \qquad G = n (p_0 s' + s_0 p') - (h_0 k' + k_0 h') .$$

7 - 3. Construction of the Points Common to a Line and a Conic Given by Five Points When the Conic is not Drawn.

Let us now find (construct) the points of intersection of the line

$$(1) \qquad e' \cdot r = e_0$$

with the conic thru the five points O, A, B, C, and D when the conic is not drawn. Its equation is given (9). e is the vector perpendicular from point O to the given line. See Fig. 7 - 1 . Multiplying (9) and (1) together we obtain:

$$(2) \qquad r \cdot L \cdot r = 0$$

$$(3) \qquad L = e_0 H - e' G$$

Splintering (2) with the Omega Proposition we obtain

$$(4) \qquad e_0 (n p' \cdot i - h' \cdot i_1) - G \cdot i_2 = G \cdot e' - e_0 (n p' \cdot s' - h' \cdot k')$$

$$(5) \qquad \Big\uparrow \begin{array}{ccc} i \wedge s & & r \\ i_1 \wedge k & & r \\ i_2 \wedge e & & r \end{array} \Big\downarrow$$

Equation (4) may be written in the compact form:

$$(6) \qquad M \cdot i = N .$$

$$(7) \qquad N = e_0 (n\acute{p} - \hat{h}) - \hat{G}$$

$$(8) \qquad N = G \cdot e' - e_0 (n p' \cdot s' - h' \cdot k') .$$

$\hat{h}$ and $\hat{G}$ are the comigrates of hand G. See the end of section 1 - L chapter one for the location of comigrates. Both M and N in equation (6) are known, M being a known vector and N a known scalar. Equation (6) is an alpha type equation and its solution for i is immediate. All reference vectors originate at point O on the conic. One simply puts a circle on M as a diameter and with O as a center and N as a radius cuts this circle in points T_1 and T_2 . This is the mechanica l solution. We do not need the analytical solution here. The lines OT_1 and OT_2 give us the directions of i . To get the directions of r from point O to the points where the given line cuts the conic we see from the Mutation Diagram that we must bisect the angles between s and i . These two angles are known since s is known and we have just determined the two directions of i namely OT_1 and OT_2. We produce the bisectors of these two angles till they meet the given line in the points where it meets the conic thru the five points O, A, B, C, and D. There will be two, one, or no points of intersection according as M is greater than, equal to, or less than N. In projective geometry this construction is accomplished by finding the double points of a projectivity.

As far as the author is aware no mechanical schemes for constructing points of intersection of a given line with a conic given by five points when the conic is not drawn have been devised except those of pure projective geometry and Mutation Geometry. For Mutation Geometry is it a histotical GO.

7 - 4. Construction of a Tangent From an External Point to a Conic Given by Five Points When the Conic is not Drawn

While we are dealing with a conic thru five points we shall show how to construct a tangent from an external point to the conic determined by the five points when the conic is not drawn. See Fig. 7 - 2 .

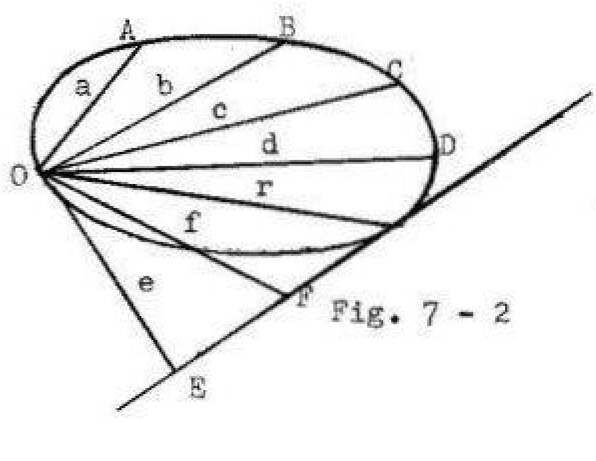

Fig. 7 - 2

Let F be the given external point and r the vector from O to the point of contact of the tangent with the conic. Let OE be the perpendicular from O to the given tangent. We designate the unit vewtor along OE by the letter e. Designate the vector OF by the letter f. One may now write the following equation:

$$(1) \qquad f \cdot e = e \cdot r$$

We now repeat equation (9) of section: (7 - 2)

$$(2) \qquad r \cdot H \cdot r = G \cdot r$$

From equations (1) and (2) one obtaind:

$$(3) \qquad r \cdot L \cdot r = 0 .$$

$$(4) \qquad L = (f \cdot e) H - e G .$$

Splintering equation (3) by means of our Omega Proposition we obtain the equation:

$$(5) \qquad (f \cdot e)(n p' \cdot j - h' \cdot j_1) - G \cdot j_2 = G \cdot e - (f \cdot e)(n p' \cdot s' - h' \cdot k') .$$

$$(6) \qquad \uparrow \begin{array}{cccc} j & \wedge & s & r \\ j_1 & \wedge & k & r \\ j_2 & \wedge & e & r \end{array} \downarrow$$

Equation (5) may now be written in the compact form:

$$(7) \qquad M \cdot j = N \cdot e$$

$$(8) \qquad M = f \cdot e (n p' - \hat{h}) - \hat{G}$$

$$(9) \qquad N = G - (n p' \cdot s' - h' \cdot k') f .$$

If we are to have one value of r we will have one value of j. The condition for this is:

$$(10) \qquad M^2 = (M . e)^2$$

$$(11) \qquad M^2 = (f . e)^2 (n p' - \hat{H})^2 + G^2 - 2 f . e \hat{G} . (n p' - \hat{H})$$

$$(12) \qquad \hat{G} . (n p' - \hat{H}) = G_0 u . e$$

where u and $(n p' - \hat{H})$ have the same magnitude and u makes the same angle with s as $(n p' - \hat{H})$ does with G. This follows from the mutation diagram since the angle that s makes with e is the same angle that G makes with $\hat{G}$. See Fig. 7 - 3.

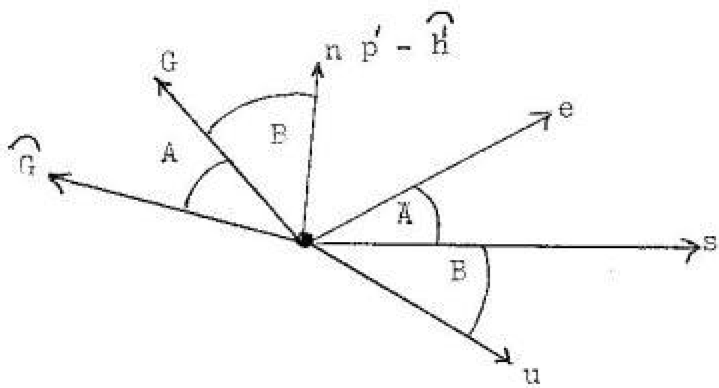

Fig. 7 - 3 .

Let A be the angle between s and e and between G and $\hat{G}$. Let B, a known angle, be the angle between G and $(n p' - \hat{H})$. Now draw vector u so that the angle between s and u is angle B then the angle between u and e is the same as the angle between $(n p' - \hat{H})$ and $\hat{G}$ namely A + B . Equation (10) then becomes:

$$(13) \qquad u^2 (f . e)^2 - 2 G_0 f . e u . e - (N . e)^2 = - G^2$$

Splintering equation (13) with our Omega Proposition we obtain

$$(14) \qquad u^2 f^2 (1 + f' . i) - 2 G_0 (u . f + f_0 u . i) + 2 G^2 = N^2 (1 + N' . i_1) .$$

(15) $\quad i \wedge f \quad e$

$\quad i_1 \wedge N \quad e$

Equation (14) may be put into prototype form:

(16) $\quad E \, . \, i = F$

(17) $\quad E = f_0 (u^2 f - 2 G_0 u) - N_0 N$

(18) $\quad F = N^2 + 2 G_0 u \, . \, f - u^2 f^2 - 2 G^2 .$

Equation (16) is a prototype equation whose solution for i is immediate either mechanically or analytically. Mechanically one simply puts a circle on E as a diameter and with one end of this diameter as a center and F as a radius cut this circle in two points determining two directions for the unit vector i . One then, according to (15) , simply bisects the angles between f and the two directions of i in order to get the two directions of e. From F the end of f one draws two perpendiculars to the two directions of the e's which are the required tangents. There will be two, one, or no tangents according as F, in eg. (16), is less than, equal to, or greater than E. In the last case point F is inside the conic.

7 - 5 A Conic Property

We shall show that the locus of the feet of the perpendiculars drawn from the focus of a conic to the tangents is a circle. This is true for all conics whether circle, parabola, ellipse, or hyperbola. See Fig. 7 - 4 .

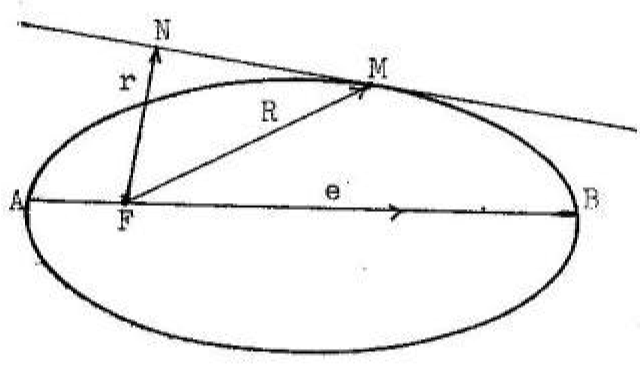

Fig. 7 - 4 .

Let F be a focus, M the point of contact of the tangent with the conic, and N the foot of the perpendicular from F. Denote FM by R and FN by r. Let e be the sensetized eccentricity along AB the diameter of the conic.

We may now write the following equations, the first being the equation of the conic, with s denoting the semi-perfolatum:

$$(1) \qquad R_o = e \cdot R + s$$

$$(2) \qquad r' \cdot R = r_o$$

By eliminating R_o from (1) and (2) one obtains:

$$(3) \qquad (e + s r_o^{-1} r') \cdot R' = 1 .$$

Differentiating (1) and replacing d R by $\breve{r}$ one obtains:

$$(4) \qquad \breve{r} \cdot R' = e \cdot \breve{r}$$

Eliminating R' from (3) and (4) one obtains the equation

$$(5) \qquad (e^2 - 1) r^2 + 2 s e \cdot r + s^2 = 0$$

which is the equation of a circle.

When e equals 1 equation (5) becomes the equation of a straight line or completing the square in (5) one gets $s / (1-e^2)$ for the radius of the circle and this becomes infinite when e is 1 which means again that the circle, in the case of the parabola, is a straight line.

In most books on analytic geometry one finds proven the statement that the perpendiculars to the tangent tangents of a parabola from a focus meet the tangents where they cross the perpendicular drawn to the parabola at the vertex. How different the ideas expressed in equation (5) which is a complete generalization for all the conics. Generalization is the mode everywhere in the New Science of Mutation Geometry.

We shall use this generalization to deal with conics when they are given by five tangents, four tangents and the point of contact on one of them, or three tangents and the points of contact on two of them.

7 - 6 A Conic Given by Five Tangents, Four Tangents and the Point of Contact on one, Three Tangents and the Point of Contact on Two of them.

Find the center of a conic and its foci when the conic is given by five tangents. See Fig. 7 - 5 .

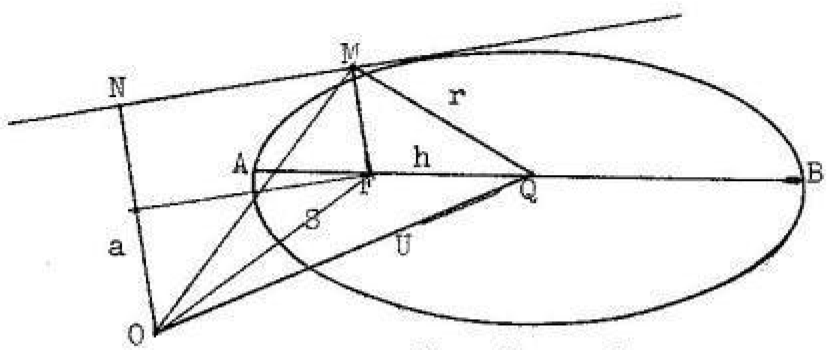

Fig. 7 - 5 .

Let a, b, c, d, f be the vectors drawn from an arbitrary origin O to the five given tangents.

Let U ans S be the vectors from O to the center and focus of the conic respectively. Let h be the vector from the center of the conic to the focus. In the Fig. we have drawn only one tangent whose perpendicular distance from O is a. Let r be the vector from the center of the conic to the point of intersection of the perpendicular from the focus F with the tangent. r then is the semi-major axis of the conic from the previous section. Recalling that r_0 is a constant, we may now write the following equations:

$$(1) \qquad r = h + (a_0 - S.a')\, a' = a + h - a' . S a'$$

$$(2) \qquad S = U + h$$

Putting (2) into (1) and squaring one obtains the relation:

$$(3) \qquad r^2 = (a + h - (U + h).\, a' a')^2$$
$$= (b + h - (U + h).\, b' b')^2.$$

Squaring and simplifying one obtains the following expression:

$$(4) \qquad (a'.U)^2 - (b'.U)^2 - (a'.h)^2 + (b'.h)^2 = 2(a-b).U + b^2 - a^2$$

Splintering the left side of (4) we obtain the expression:

$$(5) \qquad P.(a' - \widehat{b'}) = 2(a - b).U + b^2 - a^2$$

$$(6) \qquad P = (U^2 \alpha - h^2 \gamma)/2$$

$$(7) \qquad \alpha \wedge a \ U \qquad \gamma \wedge a \ h$$

We may now write three more equations analogus to (5):

$$(8) \qquad P.(a' - \widehat{c'}) = 2(a - c).U + c^2 - a^2$$

$$(9) \qquad P.(a' - \widehat{d'}) = 2(a - d).U + d^2 - a^2$$

$$(10) \qquad P.(a' - \widehat{f'}) = 2(a - f).U + f^2 - a^2.$$

Solving (5) and (8) for P we obtain:

$$(11) \qquad P = ((m_2 . U + n_2)\breve{k}_1 - (m_1 . U + n_1)\breve{k}_2)/\breve{k}_1 . k_2$$

$$k_1 = (a' - \widehat{b'}) \qquad k_2 = (a' - \widehat{c'})$$
$$k_3 = (a' - \widehat{d'}) \qquad k_4 = (a' - \widehat{f'})$$
$$m_1 = 2(a - b) \qquad m_2 = 2(a - c)$$
$$m_3 = 2(a - d) \qquad m_4 = 2(a - f)$$
$$n_1 = b^2 - a^2 \qquad n_2 = c^2 - a^2$$
$$n_3 = d^2 - a^2 \qquad n_4 = f^2 - a^2.$$

Put (11) into (9) and (10) and we obtain the two straight lines:

$$(12) \qquad L \cdot U = M$$

$$(13) \qquad H \cdot U = G.$$

$$L = (\breve{k}_1 \cdot k_3) m_2 - (\breve{k}_2 \cdot k_3) m_1 - (\breve{k}_1 \cdot k_2) m_3$$

$$H = (\breve{k}_1 \cdot k_4) m_2 - (\breve{k}_2 \cdot k_4) m_1 - (\breve{k}_1 \cdot k_2) m_4$$

$$M = (\breve{k}_1 \cdot k_2) n_3 + (\breve{k}_2 \cdot k_3) n_1 - (\breve{k}_1 \cdot k_3) n_2$$

$$G = (\breve{k}_1 \cdot k_2) n_4 + (\breve{k}_2 \cdot k_4) n_1 - (\breve{k}_1 \cdot k_4) n_2 .$$

Each piece in L, M, H, and G is constructible with a compass and straight edge. The intersection of the two lines (12) and (13) determine the vector U to the center of the conic. With U known equation (11) gives P. With the value of U and P known equation (6), since δ is known from the first part of (7), gives us the direction of γ. The second part of (7) then gives the direction of h. Equation (6) then gives the magnitude of h. h is then known. With the values of U and h known equation (2) then gives the vectors S to the foci:

$$(14) \qquad S = U \pm h .$$

This last equation gives us everything one would want to know about the conic.

The sensetized eccentricity e is given by:

$$(15) \qquad e = h / r_0$$

If the conic is a parabola the ratio h_0 / r_0 will be 1 both h_0 and r_0 being infinite in length. Also in this case the lines (12) and (13) will be parallel. If the conic is a circle h will be 0. According as h_0 / r_0 is less than or greater than one we shall have an elipse or an hyperbola.

One recalls that r_0 is the semi-major diameter of the conic and h is the distance from the center to a focus. The semi-perfoLatum is given by:

$$(16) \qquad s = (r^2 - h^2) / r_0 .$$

We have now found the most important parts of the conic: the vector U to its center, the vectors S to its foci, and the eccentricity e. One may now write its primitive equation with one focus as a refrerence point. It is

(17) $R_g = e . R + s$

where s is the semi-perfolatum. In (17) one has at his disposal the power to deal with just about any phase of the conic which suits his fancy. We may study the properties of the conic or he may perform any type of construction that is a possible one with ease. For example if one wishes to find the points of intersection of a given line with this conic given by five tangents the answer is immediate. Let the line be given by :

(18) $p . R = k$

One simply divides (17) by (18) getting the protype equation

(19) $D . R' = k$

(20) $D = s p + k e$

To find the two values of R in (19) one simply puts a circle on D as a diameter and with the focal end of D as a center and a radius k cut the circle on D as a diameter in two points. Join these two points to the focus and produce these two lines till they meet the given line in the two points where it meets the conic given by the five tangents. There will be two, one, or no solutions according as D is greater than, equal to, or less than k. The other conditions stated at the heading of this section may be dealt with in a similar fashion.

In developing this section we used the property that the perpendiculars from the focus met the tangents to the conic on the circle on the major axis as a diameter. If k in (18) should be the perpendicular to one of the tangents one would get the point of contact on the tangent out of the construction in (19). Mutation Geometry is adaptable to any previous knowledge one may have in respect to the field of geometry; to extend it and to put it in a new light.

In conventional geometry the following property of a conic is shown: See Fig. 7 - 6 . The diagonals of a quadrilateral formed by four tangents to conic have a common point with the two lines joining their points of contact. Let AB, BD, DF, and FA be the four tangents. Let M, and N be the points of contact on AF and BC.

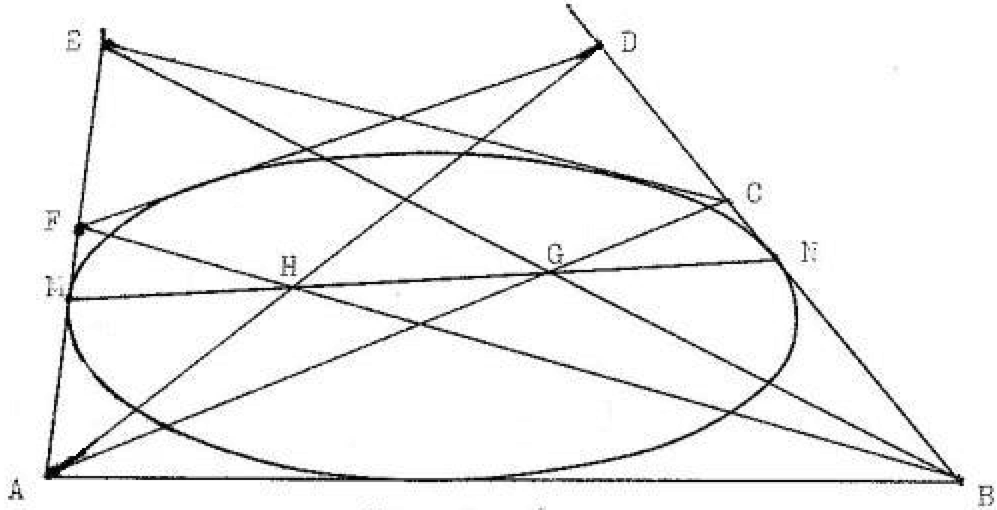

Fig. 7 - 6.

Let AD and BF meet in H which,according to the quoted proposition, is on line MN joining the points of contact M and N on tangents AF and BD respectively. If one draws a fith tangent as EC then,for the same reason,AC and BE meet in G a point on MN. If one has given five tangents to a conic then points H and G are known and the line HG determines the points of contact Mand N on the tangents AE and BD. In like fashion one may find the other points of contact on the other three tangents. It is perhaps obvious by know that when the line MN is known one may draw as many tangents to the conic as desired by drawing lines from points A and B which meet on MN and when produced cut the tangents AM and BN in points that form a tangent when joined.

When four tangents , such as AB, BD, DF, and FA and the point of contact on one of them as M on AF then one may join M to H which is known and thus the line MN is again determined.

If one has three tangents such as AB, BD, and FA and the points of contact on two of them as M and N on AB and BD then the line MN is once again determined. The line MN is determinate whether the conic is given by fife tangents, four tangents and the point of contact on one of them or by three tangents and the points of contact on two of them.

7 - 7-Tangents to a Conic From an External Point

Let it be required to construct the tangents from an external point to a conic given by five tangents, four tangents and the point of contact on one of them or by three tangents and the points of contact on two of them. See Fig. 7 - 7 below.

Let Q be the external point. Let AM, AB, and BN be three of the given tangents with M and N being the points of contact on AM and BN. Let c, d, and p be the perpendicular distances from Q to the lines AM, BN, and MN respectively. Let a and b be the vectors from Q to A and B respectively. Let QDE be one of the required tangents with AD and BE crossing MN at H. Let R, not shown in the Fig. be the vector from Q to H. Let h be the vector along the known direction NM from T_1 to H where T_1 is the foot of the perpendicular p on MN.

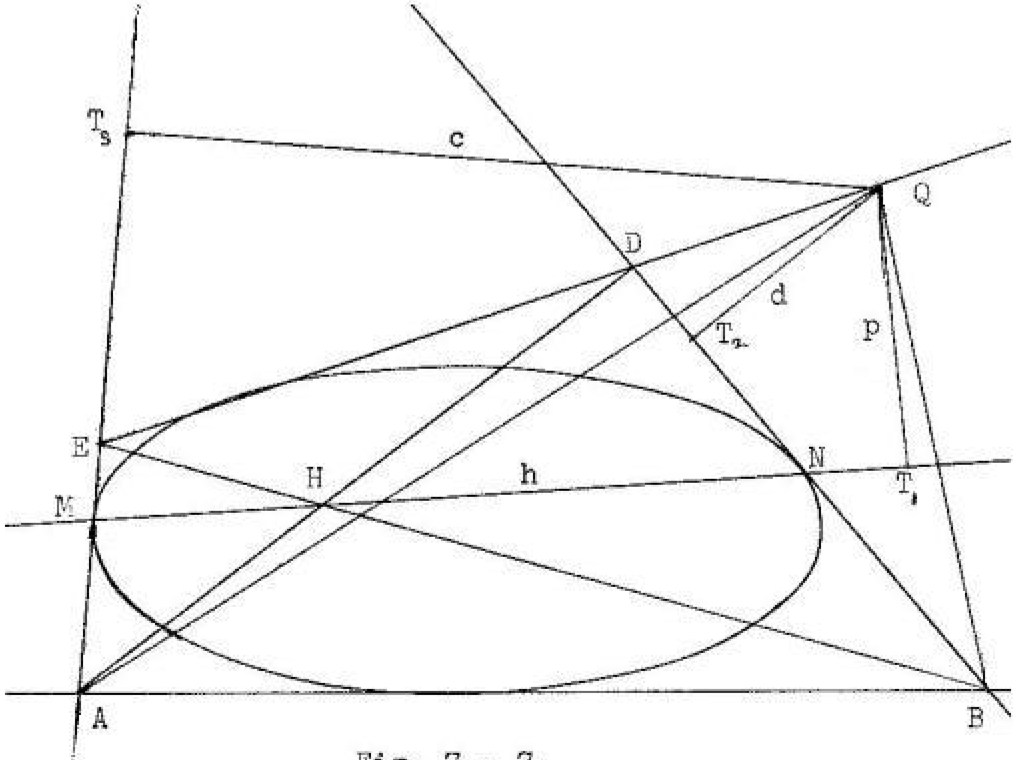

Fig. 7 - 7.

The following equations may be written:

(1) $$R = p + h.$$

The equation of the line thru the points B and H is given by:

(2) $$(R - b)^{\smile} \cdot r = b \cdot \breve{R} \quad \text{or}$$

(3) $$(g + h)^{\smile} \cdot r = b \cdot (p + h)^{\smile}$$

where $g = p - b.$

The equation of the line thru the points A and M is:

(4) $$c' \cdot r = c_0$$

The solution to (3) and (4) is :

(5) $(c_0(g + h) + b \cdot (p + h)\breve{c}) / c' \cdot (g + h) = r = QE.$

The equation of the line thru the points A and H is :

(6) $(k + h)\breve{} \cdot r = a \cdot (p + h)\breve{}$

where $k = p - a$

The equation of the line thru the points B and N is:

(7) $d' \cdot r = d_0$

The solution to (6) and (7) is :

(8) $r = QD = (d_0(k + h) + a \cdot (p - h)\breve{d}) / d' \cdot (k + h).$

QE in (5) and QD in (8) are to coincide. This is gimen by;

(9) $U \cdot V = 0.$

(10) $U = c_0(g + h)\breve{} - b \cdot (p + h)c'.$

(11) $V = d_0(k + h) + a \cdot (p + h)\breve{d}.$

Equation (9) gives a quadratic in h_0, the magnitude of h. This determines two values of h_0 and thus two locals for the point H on MN thus determining two possible tangents from Q. There will be two, one, or no tangents from Q according as the quadratic gives two, one, or no roots for h_0.

With this we bring to a close Mutation Geometry's view of the projective field. Our purpose in entering the projective field was not to study the endless properties of the conics but to show that Mutation Geometry could do the projective field with the same formulation as it used in analytic and college geometry. We are trying to illustrate the universality of the New Science of Mutation Geometry.

Beckham Martin