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Non-Linear programming

We start with systems in which the equations of constraint are linear. This condition will fit a large no. of problems of the chemical industry as we shall show later. We shall first solve a very simple problem in the geometric field. We minimize:

(1) $$S = (x_1-2)^2 + (x_2-3)^2 + (x_3-5)^2 + 12$$

Subject to the constraints:

(2) $$2x_1 + x_2 + x_3 = 12$$
(3) $$1x_1 + 2x_2 + 3x_3 = 26$$

$$x_1 > 0,\ x_2 > 0,\ x_3 > 0$$

Some point on the locus of the intersection of (2) and (3) will optimize (1).

The line of intersection of the two planes (2) and (3) has the

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direction $(1 - 5 + 3)$

and a point on this line of intersection may be found by making say $x_1 = 0$ and solving for x_2 and x_3. Doing this we get

$$0 + x_2 + x_3 = 12$$
$$0 + 2x_2 + 3x_3 = 26 \quad \text{whence}$$

$$x_2 = 10, \quad x_3 = 2$$

the equation of this line may be written:

(4) $$r = (0 + 10 + 2) + t(1 - 5 + 3) \quad \begin{pmatrix} t \geq 0 \\ t \leq 2 \end{pmatrix}$$

put (4) into (1) and we get

$$S = (t-2)^2 + (7-5t)^2 + (3t-3)^2 + 12$$

$ds/dt = 0 = 2(t-2) - 10(7-5t) + 6(3t-3)$ $\therefore$

$t = 92/70$

$r = (0 + 10 + 2) + (92/70)(1 - 5 + 3) = (92 + 240 + 416)/70$

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$$S = (92/70-2)^2+(7-92/14)^2+(276/70)^2+12 = 1.54+12 = 13.543$$

This sphere represented by (1) simply expands or contracts till it just touches the line of intersection of (2) and (3) represented by (4). Note that in this case all the components of r are positive.

Again one may solve (2) and (3) for X_2 and X_3 in terms of X_1. Doing this we get:

$$X_2 = 10-5X_1 \quad \therefore$$
$$X_3 = 2+3X_1$$

$$S = (X_1-2)^2+(7-5X_1)^2+(3X_1-3)^2+12$$

$$\partial S/\partial X_1 = 0 = 2(X_1-2)-10(7-5X_1)+6(3X-3)$$

$$X_1 = 92/70 \qquad X_2 = 24/7, \quad X_3 = 416/70$$

$$S = (92/70-2)^2+(24/7-3)^2+(416/70-5)^2+12 = 13.543$$

which is the same as before. Let us

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Change our objective function to

$$S = (x_1+2)^2+(x_2+3)^2+(x_3+5)^2+12$$

with the same constraints as before.

$$r = (0+10+2)+t(1-5+3)$$
$$= t+(10-5t)+(2+3t) \qquad \begin{pmatrix} t \geq 0 \\ t \leq 2 \end{pmatrix}$$

$$S = (t+2)^2+(13-5t)^2+(3t+7)^2+12$$

$$S' = 2(t+2)-10(13-5t)+6(3t+7) = 0$$

$$70t = 84 \quad \therefore \quad t = 1.2$$

$$r = 1.2+4+5.6$$

$$S = (3.2)^2+(7)^2+(10.6)^2+12 = 183.60 = \min$$

All the components of r are positive.

We change our objective function again:

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$$S = (x_1+2)^2 + (x_2-12)^2 + (x_3+5)^2 + 12$$

$$r = t + (10-5t) + (2+3t)$$

$$S = (t+2)^2 + (-5t-2)^2 + (3t+7)^2 + 12$$

$$S = (t+2)^2 + (5t+2)^2 + (3t+7)^2 + 12$$

$$S' = 2(t+2) + 10(5t+2) + 6(3t+7) = 0$$

$$70t = -66 \quad \therefore t \approx -0.943$$

$$r = -0.943 + 14.715 - 0.829 \quad (S = \underline{38.5})$$

and our xs are <u>not</u> all positive

If we increase t by 0.943 all our xs will be positive, giving us a new R

$$R = 0 + 10 + 2 \quad (S = \underline{69})$$

which is the smallest or first positive value on the line of intersection. The objective function S (sphere)

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will then cut, (not tangent), the line of intersection at this point and at another point either larger all positive or some negative. For this point our objective function becomes

$$S = (2)^2 + (2)^2 + (7)^2 + 12$$

$$= 4 + 4 + 49 + 12 = 69$$

which we take as the min.

One can see from the formula for S that any positive value of t greater than 0 will give a larger value of S. t must not be larger than 2 else x_2 will be negative.

Let us examine this simple system some more in detail hoping to find some facts that will be of service when dealing with

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more complex systems. We may write (2) and (3) in a more general form:

(1) $$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

(2) $$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

Let us suppose we have found two sets of solutions for the system above, and one can always do this by assigning values to one of the variables and solving for the other two, calling them $(\alpha_1, \alpha_2, \alpha_3)$, $(\gamma_1, \gamma_2, \gamma_3)$. Then we have:

(3) $$a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3 = b_1$$

(4) $$a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3 = b_2$$

(5) $$a_{11}\gamma_1 + a_{12}\gamma_2 + a_{13}\gamma_3 = b_1$$

(6) $$a_{21}\gamma_1 + a_{22}\gamma_2 + a_{23}\gamma_3 = b_2$$

Comparing (3) and (5), and (4) and (6) we get:

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(7) $$a \cdot \rho = 0$$

(8) $$A \cdot \rho = 0$$

$$a = (a_{11}, a_{12}, a_{13})$$

$$A = (a_{21}, a_{22}, a_{23})$$

$$\rho = \gamma - \delta$$

In other words ρ is orthogonal to both coefficient rows, and ρ also gives the direction of the line of intersection of (1) and (2). The equation of this line of intersection may be written:

(9) $$r = \delta + t\rho$$

where t is a scalar multiplier. r in (9) is a solution for the system (1) and (2) for it satisfies both (1) and (2). Eqs. (7) and (8) are of far reaching consequence for they

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are true for systems of many equations and many unknowns as we shall soon show.

Let us temporarily return to our original two numerical equations:

$$2x_1 + x_2 + x_3 = 12$$
$$1x_1 + 2x_2 + 3x_3 = 26$$

One solution to this system is

$$(\alpha_1, \alpha_2, \alpha_3) = (0 + 10 + 2)$$

and a second solution is

$$(\gamma_1, \gamma_2, \gamma_3) = (2 + 0 + 8)$$

$$\rho = \gamma - \alpha = 2 - 10 + 6 = 2(1 - 5 + 3)$$

$$\rho = 1 - 5 + 3$$

One can get this direction by taking column cofactors.

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It is better to take out any common factor in ρ before one finds the eq. of the line of intersection.

$$r = \alpha + t(\gamma - \alpha) = \alpha + t\rho$$

will satisfy (1) and (2) and is one solution to that system since α is a solution to the system and $(\gamma - \alpha)$ gives zero for each equation in the system.

We go to a slightly more complex system. Minimize

(1) $$S = F(x_1, x_2, x_3, x_4)$$

subject to the constraints

(2) $$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$

(3) $$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$

Let α and γ be two distinct

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Solutions ~~for~~ the system (2), (3)
then we ~~m~~ay write

(4) $$a \cdot \alpha = b_1$$
(5) $$A \cdot \alpha = b_2$$

(6) $$a \cdot \gamma = b_1$$
(7) $$A \cdot \gamma = b_2$$

comparing (4) and (6); ~~and~~ (5) and (7):

(8) $$a \cdot \rho = 0$$
(9) $$A \cdot \rho = 0$$

$$a = (a_{11}, a_{12}, a_{13}, a_{14})$$

$$A = (a_{21}, a_{22}, a_{23}, a_{24})$$

$$\rho = \gamma - \alpha$$

From (4) and (5) or (6) and (7) we get

(10) $$(b_2 a - b_1 A) \cdot \alpha = 0$$ and

(11) $$(b_2 a - b_1 A) \cdot \gamma = 0$$

which mean that all solutions to the system (2), and (3) lie in a plane thru the origin and perpendicular to the vector:

$$(b_2 a - b_1 A).$$

We revise our theory slightly:

Let us suppose that $\alpha = (\alpha_1 \cdots \alpha_n)$ is a solution to the system (2), (3) and that $\gamma = (\gamma_1 \cdots \gamma_n)$ is perpendicular to both a and A. We can then write the following four equations:

(1) $$a \cdot \alpha = b_1$$
(2) $$A \cdot \alpha = b_2$$

(3) $$a \cdot \gamma = 0$$
(4) $$A \cdot \gamma = 0$$

$\alpha + \gamma$ is also a solution to the system under consideration. Ex-

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panding the four equations we get

(5) $a_{11}\alpha_1 + a_{12}\alpha_2 + a_{13}\alpha_3 + a_{14}\alpha_4 = b_1$

(6) $a_{21}\alpha_1 + a_{22}\alpha_2 + a_{23}\alpha_3 + a_{24}\alpha_4 = b_2$

(7) $a_{11}\gamma_1 + a_{12}\gamma_2 + a_{13}\gamma_3 + a_{14}\gamma_4 = 0$

(8) $a_{21}\gamma_1 + a_{22}\gamma_2 + a_{23}\gamma_3 + a_{24}\gamma_4 = 0$

Solving (5) and (6) for α_3 and α_4 we get

(9) $\alpha_3 = [L + M\alpha_1 + N\alpha_2]$

(10) $\alpha_4 = [F + G\alpha_1 + H\alpha_2]$

Solving (7) and (8) for γ_3 and γ_4 we get

(11) $\gamma_3 = M\gamma_1 + N\gamma_2$

(12) $\gamma_4 = G\gamma_1 + H\gamma_2$

(13) $\alpha = \alpha_1 + \alpha_2 + [L + M\alpha_1 + N\alpha_2] + [F + G\alpha_1 + H\alpha_2]$

(14) $\gamma = \gamma_1 + \gamma_2 + [\ M\gamma_1 + N\gamma_2] + [\ G\gamma_1 + H\gamma_2]$

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$P = (a_{13} a_{24} - a_{14} a_{23}$

$L = (a_{24} b_1 - a_{14} b_2)/P$

$M = (a_{14} a_{21} - a_{11} a_{24})/P$

$N = (a_{14} a_{22} - a_{12} a_{24})/P$

$F = (-a_{23} b_1 + a_{13} b_2)/P$

$G = (a_{11} a_{23} - a_{13} a_{21})/P$

$H = (a_{12} a_{23} - a_{13} a_{22})/P$

We note from (13) that our solution vector d has only two unknowns in it. We are to minimize the objective function:

$$f(d_1, d_2, d_3, d_4) = f(d_1, d_2)$$

From this we have the two minimizing conditions:

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$$\partial F / \partial \delta_1 = 0$$

$$\partial F / \partial \delta_2 = 0$$

From thes two eqs. we can solve for δ_1 and δ_2 and equations (9) and (10) give us, with the help of the δ_1 and δ_2 just found, the values of δ_3 and δ_4. Our solution vector $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$ is then known and if all δ_j are positive our problem is solved.

If some of the δ_j are negative we must have this point along γ in order to maintain solution to our system and to correct δ_j. See eq. (14). We now correct our solution so that:

$$\rho = \delta + \gamma$$

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has all positive components. The Nature of the objective function F will have to play its part in this Correction, unless the number of equations in our system is 1 less than the Number of unknowns. To see this suppose we have a system of $n-1$ equations in n unknowns then.

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = 0$$
$$- \quad - \quad - \quad - \quad - \quad -$$
$$a_{n-1\,1}x_1 + a_{n-1\,2}x_2 \cdots a_{n-1\,n}x = 0$$

One can transpose the last Column on the right across the equality sign and get:

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1\,n-1}x_{n-1} = -a_{1n}x_n$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2\,n-1}x_{n-1} = -a_{2n}x_n$$
$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$
$$a_{n-1\,1}x_1 + a_{n-1\,2}x_2 + \cdots a_{n-1\,n-1}x_{n-1} = -a_{n-1\,n}x_n$$

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One can solve the square matrix on the left for $\gamma_1, \gamma_2, \ldots \gamma_{n-1}$ in terms of γ_n and get:

$$\gamma_1 = C_1 \gamma_n$$
$$\gamma_2 = C_2 \gamma_n$$
$$\cdots$$
$$\gamma_{n-1} = C_{n-1} \gamma_n .$$

Thus:

$$\gamma = \gamma_n (C_1 + C_2 + \cdots C_{n-1} + 1)$$

and thus γ has a fixed direction or $n-1$ equations in n unknowns intersect in one line or $n-1$ hyperplanes with n unknowns intersect in one line. The C_i are known quantities and constants. They are or are proportional proportional to the column cofactors in order from left to right in the original system. From (14) one sees there is an indefinite number of intersections in that system. For any

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choice of γ_1 and γ_2 one gets a definite line of intersection. From eq. (14) this choice of γ_1 and γ_2 determines the whole γ since γ_3 and γ_4 are determined by the γ_1 and γ_2, see eqs. (11) and (12).

Consider the system:

$$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 12$$
$$2\alpha_1 + 1\alpha_2 + 2\alpha_3 + 3\alpha_4 = 26$$

$$f = (\alpha_1 - 2)^2 + (\alpha_2 - 1)^2 + (\alpha_3 - 4)^2 + (\alpha_4 - 3)^2 + 12$$

$$P = 1$$

$$L = 10, \quad M = -1, \quad N = -5$$

$$F = 2, \quad G = 0, \quad H = 3$$

$$\alpha_3 = 10 - \alpha_1 - 5\alpha_2$$

$$\alpha_4 = 2 + 3\alpha_2$$

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$$f = (\alpha_1-2)^2+(\alpha_2-1)^2+(6-\alpha_1-5\alpha_2)^2+(3\alpha_2-1)^2+12$$

$$2(\alpha_1-2)-2(6-\alpha_1-5\alpha_2) = 0 = \partial f/\partial \alpha_1$$
$$2(\alpha_2-1)-10(6-\alpha_1-5\alpha_2)+6(3\alpha_2-1) = 0 = \partial f/\partial \alpha_2 \quad \therefore$$

$$4\alpha_1+10\alpha_2 = 16$$
$$12\alpha_1+70\alpha_2 = 68 \quad \therefore$$

$$\alpha_1 = 11/4, \qquad \alpha_2 = 1/2$$

and

$$\alpha_3 = 19/4, \qquad \alpha_4 = 7/2$$

putting these values into f and we get

$$f = 13\ 5/8 = 13\ 5/8 = 13.625 \text{ (Min.)}$$

To get some confirmation as to whether f is a min. we change our α_i slightly and write

$\alpha_1 = 3$, $\alpha_2 = 1/2$, $\alpha_3 = 5$, $\alpha_4 = 4$ then

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$$f_2 = (1)^2 + (1/2)^2 + (1)^2 + (1)^2 + 12 = 15.250$$

showing that f is larger for values of δ_i slightly different from those for the min.

In this case all δ_j are positive and they are the required answers. They satisfy the constraint equations.

We now consider the general case. ~~Here~~ we shall have m equations in n unknowns and we want to minimize a function $f(x_1 \dots x_n)$ subject to the m equations of constraint. We write: minimize

(1) $$f(x_1 \dots x_n)$$

Subject to:

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$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1\,n-1}x_{n-1} + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2\,n-1}x_{n-1} + a_{2n}x_n = b_2$$
(2)
$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{m\,n-1}x_{n-1} + a_{mn}x_n = b_m$$

$$n > m$$

Eq. (2) may be written:

$$a_{1\,n-m+1}x_{n-m+1} + \cdots a_{1n}x_n = b_1 - a_{11}x_1 + a_{12}x_2 - \cdots a_{1\,n-m}x_{n-m}$$
$$a_{2\,n-m+1}x_{n-m+1} + \cdots a_{2n}x_n = b_2 - a_{21}x_1 - a_{22}x_2 - \cdots a_{2\,n-m}x_{n-m}$$
(3)
$$a_{m\,n-m+1}x_{n-m+1} + \cdots a_{mn}x_n = b_m - a_{m1}x_1 - a_{m2}x_2 - \cdots a_{m\,n-m}x_{n-m}$$

The m by m matrix on the left of (3) can be solved for the x_{n-m+1} to x_n in terms of the right side of eq. (2) getting

$$x_{n-m+1} = M_0 + M_1x_1 + M_2x_2 + \cdots M_{n-m}x_{n-m}$$
(4)
$$- - - - - - - - - - - - - -$$
$$x_n = Q_0 + Q_1x_1 + Q_2x_2 + \cdots Q_{n-m}x_{n-m}$$

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These values in (4) are put into f and we get:

$$(5) \qquad f(x_1 \cdots x_n) = f(x_1 \cdots x_{n-m})$$

One then forms the set of eqs.:

$$\partial f(x_1 \cdots x_{n-m})/\partial x_1 = 0$$

$$(6) \qquad \partial f(x_1 \cdots x_{n-m})/\partial x_2 = 0$$

$$\cdots\cdots\cdots\cdots$$

$$\partial f(x_1 \cdots x_{n-m})/\partial x_{n-m} = 0$$

The system in (6) can be solved for $x_1 \cdots x_{n-m}$. Eq. (4) then, with these values set in, gives us the remaining values of r, where

$$r = x_1 + x_2 + \cdots x_n.$$

r is the minimizing vector. Our

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Corresponding δs simply omit the letters with zero subscripts in eq. (4).

With this generalization we are now in a position to discuss chemical reactions and the synthesis of desired chemical compounds, under various conditions (constraints of temp. pressure, concentration etc).

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It ought to be emphasized that one gets the coefficients of the various δ_j as a by-product in calculating the δ relation and they do not have to be calculated separately. This is quite a saving of time and effort. Keep this in mind.

In anticipation of a chemical problem of the same order we do the constraints calculation for a system of three equations in ten unknowns, the objective

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function being specified later, after looking up some quantities in published tables. We write:

$$\text{(1)}\qquad \begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots a_{110}X_{10} &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots a_{210}X_{10} &= b_2 \\ a_{31}X_1 + a_{32}X_2 + \cdots a_{310}X_{10} &= b_3 \end{aligned}$$

This may be written:

$$\begin{aligned} a_{18}X_8 + a_{19}X_9 + a_{110}X_{10} &= b_1 - a_{11}X_1 - \cdots a_{17}X_7 \\ a_{28}X_8 + a_{29}X_9 + a_{210}X_{10} &= b_2 - a_{21}X_1 - \cdots a_{27}X_7 \\ a_{38}X_8 + a_{39}X_9 + a_{310}X_{10} &= b_3 - a_{31}X_1 - \cdots a_{37}X_7 \end{aligned}$$

$$\begin{vmatrix} X_8 \\ X_9 \\ X_{10} \end{vmatrix} = \underbrace{\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}}_{P} \begin{vmatrix} b_1 - a_{11}X_1 - \cdots a_{17}X_7 \\ b_2 - a_{21}X_1 - \cdots a_{27}X_7 \\ b_3 - a_{31}X_1 - \cdots a_{37}X_7 \end{vmatrix}$$

$$X_8 = F_0 + F_1X_1 + \cdots F_7X_7$$

$$X_9 = G_0 + G_1X_1 + \cdots G_7X_7$$

$$X_{10} = H_0 + H_1X_1 + \cdots H_7X_7$$

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$$L_1 = \begin{vmatrix} a_{29} & a_{210} \\ a_{39} & a_{310} \end{vmatrix} \qquad M_1 = -\begin{vmatrix} a_{19} & a_{110} \\ a_{39} & a_{310} \end{vmatrix}$$

$$L_2 = -\begin{vmatrix} a_{28} & a_{210} \\ a_{38} & a_{310} \end{vmatrix} \qquad M_2 = \begin{vmatrix} a_{18} & a_{110} \\ a_{38} & a_{310} \end{vmatrix}$$

$$L_3 = \begin{vmatrix} a_{28} & a_{29} \\ a_{38} & a_{39} \end{vmatrix} \qquad M_3 = -\begin{vmatrix} a_{18} & a_{19} \\ a_{38} & a_{39} \end{vmatrix}$$

$$N_1 = \begin{vmatrix} a_{19} & a_{110} \\ a_{29} & a_{210} \end{vmatrix}$$

$$N_2 = -\begin{vmatrix} a_{18} & a_{110} \\ a_{28} & a_{210} \end{vmatrix}$$

$$N_3 \begin{vmatrix} a_{18} & a_{19} \\ a_{28} & a_{29} \end{vmatrix}$$

$$P = \begin{vmatrix} a_{18} & a_{19} & a_{110} \\ a_{28} & a_{29} & a_{210} \\ a_{38} & a_{39} & a_{310} \end{vmatrix}$$

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$$F_0 = (L_1 b_1 + M_1 b_2 + N_1 b_3)/P$$

$$F_1 = -(L_1 a_{11} + M_1 a_{21} + N_1 a_{31})/P$$

$$F_2 = -(L_1 a_{12} + M_1 a_{22} + N_1 a_{32})/P$$

$$F_3 = -(L_1 a_{13} + M_1 a_{23} + N_1 a_{33})/P$$

$$F_4 = -(L_1 a_{14} + M_1 a_{24} + N_1 a_{34})/P$$

$$F_5 = -(L_1 a_{15} + M_1 a_{25} + N_1 a_{35})/P$$

$$F_6 = -(L_1 a_{16} + M_1 a_{26} + N_1 a_{36})/P$$

$$F_7 = -(L_1 a_{17} + M_1 a_{27} + N_1 a_{37})/P$$

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$$G_0 = (L_2 b_1 + M_2 b_2 + N_2 b_3)/P$$

$$G_1 = -(L_2 a_{11} + M_2 a_{21} + N_2 a_{31})/P$$

$$G_2 = -(L_2 a_{12} + M_2 a_{22} + N_2 a_{32})/P$$

$$G_3 = -(L_2 a_{13} + M_2 a_{23} + N_2 a_{33})/P$$

$$G_4 = -(L_2 a_{14} + M_2 a_{24} + N_2 a_{34})/P$$

$$G_5 = -(L_2 a_{15} + M_2 a_{25} + N_2 a_{35})/P$$

$$G_6 = -(L_2 a_{16} + M_2 a_{26} + N_2 a_{36})/P$$

$$G_7 = -(L_2 a_{17} + M_2 a_{27} + N_2 a_{37})/P$$

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$$H_0 = (L_3 b_1 + M_3 b_2 + N_3 b_3)/P$$

$$H_1 = -(L_3 a_{11} + M_3 a_{21} + N_3 a_{31})/P$$

$$H_2 = -(L_3 a_{12} + M_3 a_{22} + N_3 a_{32})/P$$

$$H_3 = -(L_3 a_{13} + M_3 a_{23} + N_3 a_{33})/P$$

$$H_4 = -(L_3 a_{14} + M_3 a_{24} + N_3 a_{34})/P$$

$$H_5 = -(L_3 a_{15} + M_3 a_{25} + N_3 a_{35})/P$$

$$H_6 = -(L_3 a_{16} + M_3 a_{26} + N_3 a_{36})/P$$

$$H_7 = -(L_3 a_{17} + M_3 a_{27} + N_3 a_{37})/P$$

Our objective function is

$$F(x_1 \cdots x_{10}) = F(x_1 \cdots x_7)$$

One will then have seven equations of the type:

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$$\partial f / \partial x_5 = 0$$

from which to solve for $x_1 \cdots x_7$. We know what x_8, x_9, and x_{10} are in terms of these thus are known the min. vector r:

$$r = (x_1 \cdots x_{10})$$

The γ coefficients are the F, G, and H coefficients with F_0, G_0 and H_0 omitted. We shall presently do a numerical example.